Partial Differential Equations III

The regularity theory of De Giorgi, Nash and Moser for elliptic partial differential equations

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Content

The lecture is divided in three parts: first we recall and introduce some necessary analytic tools. Next, we tackle elliptic *equations*, using the method of De Giorgi and then that of Moser. Next, for *systems*, we give counterexamples to regularity in dimension higher than two, and show the Hölder continuity of solutions for systems with bounded coefficients.

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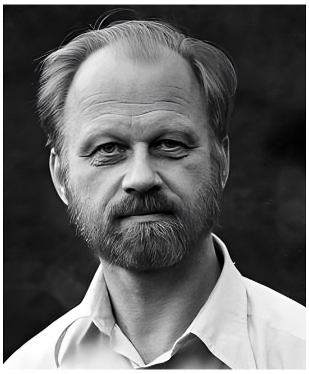
Ennio De Giorgi

1928 - 1996
Italian

De Giorgi contributions to the calculus of variations and minimal surfaces include the solution of Bernstein's problem. His solution of Hilbert's 19th in dimension larger than two was published the year before the (independent) work of Nash. It is thought that this simultaneity might be the reason why neither of them received the Fields Medal. The work of De Giorgi helped to found the then emerging field of geometric analysis.

Ennio De Giorgi on Wikipedia



Jürgen Moser

1928 - 1999
American - German - Swiss

Moser found the optimal constant in what is now known as the Moser-Trudinger inequality. Before that, he contributed to differential geometry with his work on differential forms and the problem of prescribed scalar curvature. He is also well known for revisiting the results of De Giorgi and Nash on Hilbert's 19th problem, and his introduction of the so-called Moser iteration. He was awarded the Wolf Prize in 1995 for his work on nonlinear PDEs and the stability of Hamiltonian systems.

Jürgen Moser on Wikipedia

John Nash

1928 - 2015
American

Nash is best known for his work on game theory and the notion of Nash equilibrium, which earned him the Nobel Prize in Economics in 1994. But his most significant mathematical works are his embedding theorem for Riemannian manifolds and his regularity theorems for parabolic and elliptic equations, which is the subject of this lecture. The latter was his last significant work before a long period of mental illness. He died in 2015, just a few days after receiving the Abel Prize.

John Nash on Wikipedia

Not written with ⟨popular large language model⟩, which believes De Giorgi was awarded the Fields Medal in 1990 at the age of 62.

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1 Motivation

Shortly after his talk on the subject at the 2nd International Congress of Mathematicians (ICM) in 1900, HILBERT exposed his famous list of 23 problems. Among those, the 19th (*Are the solutions of regular variational problems always analytic?*) broadly raises the question of the regularity of solutions to certain partial differential equations. Here is how he introduces the problem:

> *Eine der begrifflich merkwürdigsten Thatsachen in den Elementen der Theorie der analytischen Funktionen erblicke ich darin, daß es partielle Differentialgleichungen giebt, deren Integrale sämtlich notwendig analytische Funktionen der unabhängigen Variabeln sind, die also, kurz gesagt, nur analytischer Lösungen fähig sind.*

which is translated by Mary Frances WINSTON NEWSON[1] as follows:

> *One of the most remarkable facts in the elements of the theory of analytic functions appears to me to be this: that there exist partial differential equations whose integrals are all of necessity analytic functions of the independent variables, that is, in short, equations susceptible of none but analytic solutions.*

He says more precisely that there is a class of partial differential equations (citing among others the Laplace equation, Liouville's equation and the minimal surface equation) which have only analytic solutions, and that most of these equations are in fact the Euler-Lagrange equations for what he calls *regular variational problems*. They are of the form

$$\min_u \int F(\nabla u(x), u(x), x)\, \mathrm{d}x\,,$$

where F is analytic and D^2F is positive definite, which means the corresponding Euler-Lagrange equation is elliptic. HILBERT asks if the solutions to the Euler-Lagrange equations associated to such problems are necessarily analytic, even when requiring the solutions to meet boundary conditions, which are not analytic themselves (but only continuous).

In dimension two, BERNSTEIN has shown in 1904 that C^3 solutions are in fact analytic. This result is further refined by LICHTENSTEIN (1912), who requires C^2 solutions, and by HOPF (1929), who requires $C^{1,\alpha}$. After that, MORREY completely solves the two dimensional problem in 1938.

The ideas of MORREY do not apply in higher dimensions, and there, a gap remains: the *direct method* in the calculus of variations solves the problem of the existence of solutions, but only in the class of *weak* solutions, *i.e.* in $W^{1,2}$. This can be improved to $W^{2,2}$ regularity by using the ellipticity condition, but the argument cannot be iterated further.

In this lecture, we deal with the key result of DE GIORGI and NASH, who proved that in the case where F depends only on ∇u, $W^{1,2}$ extrema in fact have HÖLDER continuous first derivative in the interior. Existing SCHAUDER type arguments are then enough to conclude to analycity of solutions.

This was later generalized to other settings, including integrands F depending also on x and u and nonlinear equations. LADYZHENSKAYA and URAL'TSEVA have also shown regularity up to the boundary. Other notable contributions where made by GIUSTI, GIAQUINTA and MIRANDA.

[1](1869 - 1959), American mathematician. She was the first to translate and publish HILBERT's problems into English.

2 Some functional analytic tools

Hölder spaces

Definition 2.1 (Continuous functions). *For a set $\Omega \subset \mathbb{R}^n$ we define*

(i) $C^0(\Omega, \mathbb{R}^m) = C(\Omega, \mathbb{R}^m)$ *as the set of all continuous functions* $f : \Omega \to \mathbb{R}^m$*;*

(ii) $C^0(\overline{\Omega}, \mathbb{R}^m) = C(\overline{\Omega}, \mathbb{R}^m)$ *as the set of all functions in* $C(\Omega, \mathbb{R}^m)$ *which can be continuously extended to the closure* $\overline{\Omega}$ *of* Ω*.*

A function $f \in C(\Omega, \mathbb{R}^m)$ is not necessarily bounded on Ω. But if f is bounded and uniformly continuous, then it can be uniquely extended up to the boundary, hence, it can actually be considered as a function in $C(\overline{\Omega}, \mathbb{R}^m)$.

The higher-order spaces C^k are defined accordingly. In particular $C_c^\infty(\Omega, \mathbb{R}^m)$ denotes the space of smooth functions with compact support in Ω.

Definition 2.2 (Hölder semi-norm). *Let $0 < \alpha \leqslant 1$, $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$. The α-Hölder semi-norm of f in S is given by*

$$[f]_{C^{0,\alpha}(S,\mathbb{R}^m)} := \sup_{\substack{x,y \in S \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} .$$

Definition 2.3. *Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$.*

(i) $C^{0,\alpha}(\Omega, \mathbb{R}^m)$ *is the set of all functions* $f \in C(\Omega, \mathbb{R}^m)$ *such that, for every compact set* $K \subset \Omega$, $[f]_{C^{0,\alpha}(K,\mathbb{R}^m)}$ *is finite.*

(ii) $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ *is the set of all bounded functions* $f \in C(\overline{\Omega}, \mathbb{R}^m)$ *such that* $[f]_{C^{0,\alpha}(\overline{\Omega},\mathbb{R}^m)}$ *is finite.*

Remark 2.4. *We stress that there are different conventions for the definition of Hölder spaces. These are sometimes introduced as the spaces of bounded functions which are* uniformly *α-Hölder continuous in Ω.*

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. The spaces $C^k(\overline{\Omega}, \mathbb{R}^m)$ and $C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ are Banach spaces, equipped with the norms*

$$\|f\|_{C^k(\overline{\Omega},\mathbb{R}^m)} := \sum_{0 \leqslant |\beta| \leqslant k} \sup_{x \in \overline{\Omega}} |D^\beta f(x)| ,$$

$$\|f\|_{C^{k,\alpha}(\overline{\Omega},\mathbb{R}^m)} := \|f\|_{C^k(\overline{\Omega},\mathbb{R}^m)} + \sum_{|\beta|=k} \sup_{x \in \overline{\Omega}} [D^\beta f]_{C^{0,\alpha}(\overline{\Omega},\mathbb{R}^m)} .$$

Remark 2.6.

- $C^{0,\alpha}$ *is not separable, consider for example* $x \mapsto |x - x_0|^\alpha$ *for* $x_0 \in [0,1]$*. Smooth functions are not dense in* $C^{0,\alpha}$

- *If $0 < \alpha_1 \leqslant \alpha_2 \leqslant 1$ and $k \in \mathbb{N}$, we have the embeddings*

  $$C^{k,1}(\overline{\Omega}, \mathbb{R}^m) \subset C^{k,\alpha_2}(\overline{\Omega}, \mathbb{R}^m) \subset C^{k,\alpha_1}(\overline{\Omega}, \mathbb{R}^m) \subset C^k(\overline{\Omega}, \mathbb{R}^m) .$$

  However, the inclusions $C^{k+1}(\overline{\Omega}, \mathbb{R}^m) \subset C^{k,1}(\overline{\Omega}, \mathbb{R}^m)$ *can fail, depending on* Ω*.*

Sobolev spaces

Definition 2.7 (Weak derivative). *Let $\Omega \subset \mathbb{R}^n$ open, $1 \leqslant p \leqslant \infty$ and let $\beta \in \mathbb{N}^n$ be a multiindex. We say that $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ has a β-th weak (or distributional) partial derivative in $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ if there exists a function $g_\beta =: D^\beta f$ in $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that for every test function $\phi \in C^\infty_c(\Omega, \mathbb{R}^m)$ we have*

$$\int_\Omega \langle f, D^\beta \phi \rangle \,\mathrm{d}x = (-1)^{|\beta|} \int_\Omega \langle g_\beta, \phi \rangle \,\mathrm{d}x \,.$$

If for some $k > 0$, the β-th weak partial derivatives of f exist in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for all multiindices β with $0 \leqslant |b| \leqslant k$, we say that f is weakly differentiable up to order *k.*

When it exists, the weak derivative is unique, up to a subset of measure zero.

Definition 2.8 (Sobolev spaces). *Let $\Omega \in \mathbb{R}^n$ open, $k > 0$.*

- *Let $1 \leqslant p \leqslant \infty$. We call the* Sobolev space *and write $W^{k,p}(\Omega, \mathbb{R}^m)$ the set of functions $f \in L^p(\Omega, \mathbb{R}^m)$ such that the weak derivatives $D^\beta f$ exist in $L^p(\Omega, \mathbb{R}^m)$ for all multiindices β with $0 \leqslant |\beta| \leqslant k$. This space is endowed with the norm*

$$\|f\|_{W^{k,p}(\Omega,\mathbb{R}^m)} := \begin{cases} \left(\sum_{0\leqslant|b|\leqslant k} \|D^\beta f\|^p_{L^p(\Omega,\mathbb{R}^m)}\right)^{\frac{1}{p}} & \text{if } 1 \leqslant p \leqslant \infty \\ \sum_{0\leqslant|b|\leqslant k} \|D^\beta f\|_{L^\infty(\Omega,\mathbb{R}^m)} & \text{if } p = \infty \,. \end{cases}$$

- *For $1 \leqslant p < \infty$, we denote by $W^{k,p}_0(\Omega, \mathbb{R}^m)$ the closure of $C^\infty_c(\Omega, \mathbb{R}^m)$ in $W^{k,p}(\Omega, \mathbb{R}^m)$,* i.e.

$$W^{k,p}_0(\Omega, \mathbb{R}^m) := \left\{ f \in W^{k,p}(\Omega, \mathbb{R}^m) : \text{ there exists } (f_j)_{j\in\mathbb{N}} \subset C^\infty_c(\Omega, \mathbb{R}^m) \text{ with } f_j \to f \text{ in } W^{k,p}(\Omega, \mathbb{R}^m) \right\} \,.$$

Endowed with their respective norms, the spaces $W^{k,p}(\Omega, \mathbb{R}^m)$ are Banach spaces for all $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}_{>0}$.

Lemma 2.9 (Weak differentiability via classical derivatives on large sets, [1, Lemma 1.41]). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider $f \in L^q(\Omega) \cap C^1(\Omega \setminus E)$ with $Df \in L^p(\Omega \setminus E, \mathbb{R}^n)$ for some $1 \leqslant p \leqslant q \leqslant \infty$ and some subset $E \subset \Omega$. If E satisfies*

$$\inf \left\{ \|\psi\|_{W^{1,q'}(\mathbb{R}^n,[0,1])} : \psi \in C^\infty_c(\mathbb{R}^n) \text{ with } \psi \geqslant \mathbb{1}_E \right\} = 0 \,, \tag{2.1}$$

then we have $f \in W^{1,p}(\Omega)$ and its weak derivative Df coincides almost everywhere with the classical derivative.

Proof. We fix a test function $\phi \in C^\infty_c(\Omega)$ and a coordinate direction $1 \leqslant i \leqslant n$. By assumption, we can choose a sequence of functions $(\psi_j)_{j\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n, [0,1])$ such that (w.l.o.g.) $\psi_j \geqslant \mathbb{1}_{U_j}$ for all $j \in \mathbb{N}$, where U_j is a neighbourhood of E, and such that

$$\|\psi_j\|_{W^{1,q'}(\mathbb{R}^n)} \to 0 \quad \text{ and } \quad \psi_j(x) \to 0 \text{ for a.e. } x \in \mathbb{R}^n \,,$$

as $j \to \infty$. Then, since $\phi(1 - \psi_j) \in C^\infty_c(\Omega \setminus E)$, we can integrate by parts:

$$\lim_{j\to\infty} \int_\Omega f\, D_i \left(\phi(1-\psi_j)\right) \,\mathrm{d}x = -\lim_{j\to\infty} \int_\Omega D_i f\, \phi(1-\psi_j) \,\mathrm{d}x = -\int_\Omega D_i f \phi \,\mathrm{d}x \,,$$

by dominated convergence. We have also used $D_i f \phi \in L^p(\Omega)$, with $D_i f$ extended by 0 to all of Ω. Thus, we have

$$\begin{aligned} \int_\Omega f D_i \phi \,\mathrm{d}x &= \lim_{j\to\infty} \int_\Omega f D_i \left(\phi(1-\psi_j) + \phi\psi_j\right) \,\mathrm{d}x \\ &= -\int_\Omega D_i f \phi \,\mathrm{d}x + \lim_{j\to\infty} \int_\Omega f \left(D_i\phi\psi_j + \phi D_i \psi_j\right) \,\mathrm{d}x \,. \end{aligned}$$

The last term vanishes since $\lim_{j\to\infty} \|\psi_j\|_{W^{1,q'}(\mathbb{R}^n)} = 0$ and $f\phi$, $fD_i\phi \in L^q(\Omega)$, by Hölder's inequality. In other words, Df satisfies the integration by parts formula, *i.e.* Df is the weak derivative of f on Ω. Thus, we have shown $f \in W^{1,p}(\Omega)$. □

Remark 2.10.

- *The condition* (2.1) *means that E is of vanishing* $W^{1,q'}$-capacity.
- *In general, classical differentiability outside of a null set is not enough. The* Cantor *function is a example.*

Morrey and Campanato spaces Here, we introduce the Morrey and Campanato spaces, which are subspaces of L^p-spaces, with growth conditions on the norm over small balls. When dealing with the regularity of weak solutions, we do not have access to pointwise values of the solutions but only to norms and other integral quantities. These spaces offer a finer degree of control and somehow bridge between the "weaker" L^p spaces and the "stronger" $C^{m,\alpha}$ spaces.

In the following, $\Omega \subset \mathbb{R}^n$ is open, $p \in [1, \infty)$ and $\lambda \geqslant 0$.

Definition 2.11 (Morrey space)**.** *We call the* Morrey space *and denote by* $L^{p,\lambda}(\Omega, \mathbb{R}^m)$ *the set of all functions* $f \in L^p(\Omega, \mathbb{R}^m)$ *such that*

$$\|f\|^p_{L^{p,\lambda}(\Omega,\mathbb{R}^m)} := \sup_{x_0\in\overline{\Omega},\, \rho>0} \min\{\rho, 1\}^{-\lambda} \int_{B_\rho(x_0)\cap\Omega} |f|^p \,\mathrm{d}x$$

is finite.

Definition 2.12 (Campanato space)**.** *We call the* Campanato space *and denote by* $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^m)$ *the set of all functions* $f \in L^p(\Omega, \mathbb{R}^m)$ *such that*

$$[f]^p_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^m)} := \sup_{x_0\in\overline{\Omega},\, \rho>0} \rho^{-\lambda} \int_{B_\rho(x_0)\cap\Omega} |f - f_{B_{\rho(x_0)}\cap\Omega}|^p \,\mathrm{d}x$$

is finite.

Here, we used the notation $f_A = \fint_A f \,\mathrm{d}x = |A|^{-1} \int_A f \,\mathrm{d}x$.

Remark 2.13.

- *We will mostly be interested in bounded domains* Ω*. In this case, the factor* $\min\{\rho^{-\lambda}, 1\}$ *in Definition* 2.11 *is typically replaced by* $\rho^{-\lambda}$.
- *Because these spaces belong to* L^p*, the finiteness of the supremum in the definition only really matters for small* ρ, i.e. $\rho < \rho_0$*, for some fixed, positive* ρ_0.

In other words, The Morrey space $L^{p,\lambda}$ is a subset of L^p containing functions f on a domain $\Omega \in \mathbb{R}^n$ such that the integral of $|f|^p$ over a ball B of radius ρ centered at x_0 goes to zero at least as fast as ρ^λ uniformly in x_0. The purpose of the restriction of L^p to the subspace $L^{p,\lambda}$ is that it allows for an alternative to the Sobolev embedding theorem which can be used with exponent p adapted to the equation (and not, for example, fixed by dimensionality). This allows "trading p for λ": choosing a smaller p results in having to show a Morrey property with larger λ, see Remark 2.15 below.

The Campanato space $\mathcal{L}^{p,\lambda}$ is defined similarly as the corresponding Morrey space, but the integral of $|f|^p$ is replaced by the integral of $|f - f_B|^p$. This is less restrictive, so $L^{p,\lambda} \subset \mathcal{L}^{p,\lambda}$. The less restrictive definition of Campanato spaces allows for an integral characterization of Hölder continuity by Campanato's theorem: $C^{0,\alpha} \cong \mathcal{L}^{p,n+p\alpha}$, if Ω is bounded, with a regular enough boundary.

Although this will not really be needed, we note that for $1 \leqslant p < \infty$ and $\lambda \geqslant 0$, $L^{p,\lambda}$ and $\mathcal{L}^{p,\lambda}$ are Banach spaces, when endowed with the norms $\|\cdot\|_{L^{p,\lambda}}$ and $[\,\cdot\,]_{\mathcal{L}^{p,\lambda}} + \|\cdot\|_{L^p}$, respectively.

More can be said in the case where Ω is bounded and regular enough, in the sense that it has the following property:

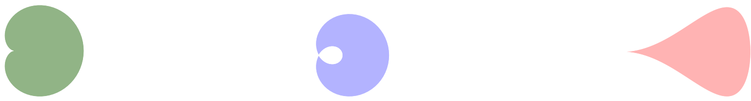

Figure 1: Which domains meet AHLFORS' condition? Which domains have a LIPSCHITZ boundary?

Definition 2.14 (AHLFORS'[2] regularity condition)**.** *An open, bounded set $\Omega \subset \mathbb{R}^n$ is said to satisfy AHLFORS' regularity condition if*

$$|\Omega \cap B_\rho(x_0)| \geqslant A\rho^n \quad \text{for all } x_0 \in \overline{\Omega} \text{ and every } \rho \leqslant \operatorname{diam}(\Omega)\,, \tag{2.2}$$

for some constant $A > 0$. Note that the left-hand side is always bounded from above by $\omega_n \rho^n$, where $\omega_n = |B_1|$.

This condition forbids external cusps, and is in particular satisfied for LIPSCHITZ domains. It also allows one to replace the factor $\rho^{-\lambda}$ by $|\Omega \cap B_\rho(x_0)|^{-\lambda/n}$ in the definitions of the MORREY and CAMPANATO spaces.

To get a bit more intuition about these spaces, we highlight the following equivalences:

Remark 2.15. *Let $\Omega \subset \mathbb{R}^n$ be bounded and such that (2.2) holds.*

- *For $0 \leqslant \lambda < n$, we have $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^m) = L^{p,\lambda}(\Omega, \mathbb{R}^m)$. We also have*

$$\mathcal{L}^{p,0}(\Omega, \mathbb{R}^m) = L^{p,0}(\Omega, \mathbb{R}^m) = L^p(\Omega, \mathbb{R}^m)\,.$$

- *For $\lambda = n$, the spaces $L^{p,n}(\Omega, \mathbb{R}^m)$ are all equivalent and coincide with $L^\infty(\Omega, \mathbb{R}^m)$. For $\Omega = Q_0$, where Q_0 is a n-cube, the space $\mathcal{L}^{1,n}(Q_0, \mathbb{R}^m)$ coincides with the space $\mathrm{BMO}(Q_0, \mathbb{R}^m)$, which we define below. From the remark below we have $\mathcal{L}^{p,n}(Q_0, R^m) \subset \mathcal{L}^{1,n}(Q_0, R^m) \cong \mathrm{BMO}(Q_0, R^m)$. Actually, we can prove that the reverse inclusion holds, so that $\mathcal{L}^{p,n}(Q_0, \mathbb{R}^m) \cong \mathrm{BMO}(Q_0, \mathbb{R}^m)$ for all $1 \leqslant p < \infty$.*

- *For $\lambda > n$, we have essentially $L^{p,\lambda}(\Omega, \mathbb{R}^m) \cong \{0\}$. For $n < \lambda \leqslant n + p$, the spaces $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^m)$ offer an integral charaterization of HÖLDER continuous functions (see below). For $\lambda > n + p$ and Ω connected, we have $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^m) \cong \{constants\}$.*

Remark 2.16 (Inclusions of MORREY and CAMPANATO spaces)**.**

- *Assume that Ω is bounded. Recall that $L^q(\Omega) \subset L^p(\Omega)$ for $1 \leqslant p \leqslant q \leqslant \infty$. In particular, for any ball $B_\rho(x_0) \subset \mathbb{R}^n$ and $f \in L^q(B_\rho(x_0))$, we have*

$$\fint_{B_\rho(x_0)} f^p \,\mathrm{d}x = \fint_{B_\rho(x_0)} f^{q\frac{p}{q}} \,\mathrm{d}x \leqslant \left(\fint_{B_\rho(x_0)} f^q \right)^{\frac{p}{q}}$$

by JENSEN's inequality, from which it follows

$$\|f\|_{L^p(B_\rho(x_0))} \leqslant c(n)\rho^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^q(B_\rho(x_0))}\,,$$

where we have used $|B_\rho(x_0)| = c_1(n)\rho^n$.

- *Assume additionally that Ω fulfills AHLFORS' condition Equation (2.2). As noted above, this allows us to replace $\rho^{-\lambda}$ by $|\Omega \cap B_\rho(x_0)|^{-\frac{\lambda}{n}}$ in Definitions 2.11 and 2.12. Applying the above to $f \in L^{q,\mu}(\Omega)$, we get*

$$\begin{aligned} |\Omega \cap B_\rho(x_0)|^{-\frac{\lambda}{n}} \int_{B_\rho(x_0)\cap\Omega} f^p \,\mathrm{d}x &\leqslant c(n)^p \left(|\Omega \cap B_\rho(x_0)|^{-\frac{q\lambda}{np}+q\left(\frac{1}{p}-\frac{1}{q}\right)} \int_{B_\rho(x_0)\cap\Omega} f^q \,\mathrm{d}x \right)^{\frac{p}{q}} \\ &\leqslant c(n)^p \left(|\Omega \cap B_\rho(x_0)|^{-\frac{q\lambda}{np}+q\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\mu}{n}} |\Omega \cap B_\rho(x_0)|^{-\frac{\mu}{n}} \int_{B_\rho(x_0)\cap\Omega} f^q \,\mathrm{d}x \right)^{\frac{p}{q}}, \end{aligned}$$

[2] Lars AHLFORS (1907-1996), Finnish, was awarded the first FIELDS Medal in 1936.

where the right-hand side is bounded if $-\frac{q\lambda}{np} + q\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{\mu}{n} \geqslant 0$*. It follows*

$$L^{q,\mu}(\Omega) \subset L^{p,\lambda}(\Omega) \quad \textit{and} \quad \mathcal{L}^{q,\mu}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$$

whenever

$$q \geqslant p \quad \textit{and} \quad \frac{n-\lambda}{p} \geqslant \frac{n-\mu}{q}$$

hold.

Among these domains, domains with Lipschitz boundary are important, in the sense that they have the *extension property*.

Proposition 2.17. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded, such that* $\partial\Omega$ *is* Lipschitz*. Then* Ω *has the* extension property, i.e. *for all* $1 \leqslant p < \infty$ *and any open set* $\widetilde{\Omega} \Supset \Omega$*, there exists* $c = c(n, \Omega, \Omega')$ *such that for all* $u \in W^{1,p}(\Omega)$*, there exists* $\tilde{u} \in W^{1,p}(\widetilde{\Omega})$ *such that* $\tilde{u}|_{\Omega} = u$ *and*

$$\|\tilde{u}\|_{W^{1,p}(\widetilde{\Omega})} \leqslant c\|u\|_{W^{1,p}(\Omega)}\,.$$

Theorem 2.18 (Campanato)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded and satifying* Ahlfors*' regularity condition* (2.2) *for some* $A > 0$*. Then, for every* $0 < \alpha \leqslant 1$ *and* $1 \leqslant p < \infty$*, we have the isomorphy*

$$\mathcal{L}^{p,n+p\alpha}(\Omega, \mathbb{R}^m) \cong C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m)\,,$$

and the semi-norms $[\,\cdot\,]_{\mathcal{L}^{p,n+p\alpha}(\Omega,\mathbb{R}^m)}$ *and* $[\,\cdot\,]_{C^{0,\alpha}(\overline{\Omega},\mathbb{R}^m)}$ *are equivalent.*

Proof. We follow [11].

We consider only the case $m = 1$, the general case follows by considering components individually.

- **Step 1:** $C^{0,\alpha}(\overline{\Omega}) \subset \mathcal{L}^{p,n+p\alpha}(\Omega)$

  Let $f \in C^{0,\alpha}(\overline{\Omega})$. We have that for every $x_0 \in \overline{\Omega}$ and $\rho > 0$, there exists $y \in \Omega \cap B_\rho(x_0)$ such that $f(y) = f_{\Omega\cap B_\rho(x_0)}$. It then holds:

$$\begin{aligned}
\int_{\Omega\cap B_\rho(x_0)} |f(x) - f_{\Omega\cap B_\rho(x_0)}|^p \,\mathrm{d}x &= \int_{\Omega\cap B_\rho(x_0)} |f(x) - f(y)|^p \,\mathrm{d}x \\
&= \int_{\Omega\cap B_\rho(x_0)} \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p}} |x-y|^{\alpha p} \,\mathrm{d}x \\
&\leqslant |\Omega \cap B_\rho(x_0)|\, [f]^p_{C^{0,\alpha}(\overline{\Omega})} (2\rho)^{p\alpha} \\
&\leqslant c(n,p)[f]^p_{C^{0,\alpha}(\overline{\Omega})} \rho^{n+p\alpha}\,.
\end{aligned}$$

  This shows the following bound for the semi-norm:

$$[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} \leqslant c(n,p)[f]_{C^{0,\alpha}(\overline{\Omega})}\,.$$

  Since Ω is bounded, we also have a bound on the L^p norm:

$$\|f\|_{L^p(\Omega)} \leqslant |\Omega|^{1/p} \|f\|_{C^0(\overline{\Omega})}\,.$$

  By combining these two estimates, we get

$$\|f\|_{\mathcal{L}^{p,n+p\alpha}(\Omega)} \leqslant c(n,p,\Omega)\|f\|_{C^{0,\alpha}(\overline{\Omega})}\,.$$

- **Step 2:** Continuous representative for functions in $\mathcal{L}^{p,n+p\alpha}(\Omega)$.

Let $f \in \mathcal{L}^{p,n+p\alpha}(\Omega)$. We first need an estimate for the average of CAMPANATO functions on balls. We take $x_0 \in \overline{\Omega}$ and $0 < r < R \leqslant \operatorname{diam}(\Omega)$, and we use the JENSEN inequality to compute

$$\begin{aligned}
|f_{\Omega\cap B_r(x_0)} - f_{\Omega\cap B_R(x_0)}| &\leqslant \left(⨍_{\Omega\cap B_r(x_0)} |f(x) - f_{\Omega\cap B_R(x_0)}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leqslant |\Omega \cap B_r(x_0)|^{-\frac{1}{p}} R^{\frac{n}{p}+\alpha} \left(R^{-n-p\alpha} \int_{\Omega\cap B_R(x_0)} |f(x) - f_{\Omega\cap B_R(x_0)}|^p\right)^{\frac{1}{p}} \\
&\leqslant c(n,p,A) r^{-\frac{n}{p}} R^{\frac{n}{p}+\alpha} [f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)}\,, \qquad (2.3)
\end{aligned}$$

where we used AHLFORS' condition in the last step. We now consider the sequence $\left(f_{\Omega\cap B_{r_j}(x_0)}\right)_{j\in\mathbb{N}}$, where $r_j = 2^{-j}R$ for some $0 < R \leqslant \operatorname{diam}(\Omega)$. Thanks to (2.3), for $0 \leqslant j < h$, we have:

$$\begin{aligned}
|f_{\Omega\cap B_{r_h}(x_0)} - f_{\Omega\cap B_{r_j}(x_0)}| &\leqslant \sum_{j\leqslant \ell\leqslant h-1} |f_{\Omega\cap B_{r_{\ell+1}}(x_0)} - f_{\Omega\cap B_{r_\ell}(x_0)}| \\
&\leqslant c(n,p,A)\,[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} R^\alpha \sum_{j\leqslant \ell\leqslant h-1} 2^{(\ell+1)\frac{n}{p}} 2^{-\ell(\frac{n}{p}+\alpha)} \\
&\leqslant c(n,p,\alpha,A)\,[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} R^\alpha 2^{-j\alpha} \\
&= c(n,p,\alpha,A)\,[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} r_j^\alpha\,, \qquad (2.4)
\end{aligned}$$

where we underline that this estimate is independent of x_0. So, the sequence of averages is not only a CAUCHY sequence, which yields pointwise convergence to some $f^*(x_0)$, it also converges *uniformly* over $\overline{\Omega}$ to f^*. Also note that due to Theorem A.1, f^* is also a representative of f in $\mathcal{L}^{p,n+p\alpha}(\Omega)$. Moreover, for fixed r, the function $x \mapsto f_{\Omega\cap B_r(x)}$ is continuous, so that f^* is continuous as the uniform limit of a sequence of continuous functions. This is the representative we are looking for.

- **Step 3:** HÖLDER continuity of f^*.

  We now take two point $x \neq y \in \overline{\Omega}$ and set $r := |x-y| > 0$. Then, we have

$$|f^*(x) - f^*(y)| \leqslant |f^*(x) - f_{\Omega\cap B_{2r}(x)}| + |f_{\Omega\cap B_{2r}(x)} - f_{\Omega\cap B_{2r}(y)}| + |f_{\Omega\cap B_{2r}(y)} - f^*(y)|\,. \qquad (2.5)$$

Passing to the limit $h \to \infty$ in (2.4), the first and third terms on the right-hand side can be estimated by

$$c(n,p,\alpha,A)[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)}|x-y|^\alpha\,,$$

so, in order to bound $[f^*]_{C^{0,\alpha}(\overline{\Omega})}$, only the second term needs to be dealt with. We do this as follows: first recall $r = |x-y|$ and note that

$$\Omega \cap B_{2r}(x) \cap B_{2r}(x) \supset (\Omega \cap B_r(x)) \cup (\Omega \cap B_r(y))\,.$$

With HÖLDER's inequality and the fact that $f \in \mathcal{L}^{p,n+p\alpha}(\Omega)$ we compute:

$$\begin{aligned}
|f_{\Omega\cap B_{2r}(x)} - f_{\Omega\cap B_{2r}(y)}| &\leqslant ⨍_{\Omega\cap B_{2r}(x)\cap B_{2r}(y)} \left(|f_{\Omega\cap B_{2r}(x)} - f(z)| + |f(z) - f_{\Omega\cap B_{2r}(y)}|\right) \mathrm{d}z \\
&\leqslant \quad |\Omega\cap B_r(x)|^{-1} |\Omega\cap B_{2r}(x)|^{\frac{p-1}{p}} \left(\int_{\Omega\cap B_{2r}(x)} |f_{\Omega\cap B_{2r}(x)} - f(z)|^p \,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\quad + |\Omega\cap B_r(y)|^{-1} |\Omega\cap B_{2r}(y)|^{\frac{p-1}{p}} \left(\int_{\Omega\cap B_{2r}(y)} |f_{\Omega\cap B_{2r}(y)} - f(z)|^p \,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\leqslant c(n,A)[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} r^{-n+n\frac{p-1}{p}+\frac{n+p\alpha}{p}} \\
&= c(n,A)[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} r^\alpha\,.
\end{aligned}$$

All together we get

$$|f^*(x) - f^*(y)| \leqslant c(n,p,\alpha,A)[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)}|x-y|^\alpha .$$

Since $x, y \in \overline{\Omega}$ are arbitrary, we have in fact

$$[f^*]_{C^{0,\alpha}(\overline{\Omega})} \leqslant c(n,p,\alpha,A)[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} , \tag{2.6}$$

which proves that the norms are indeed equivalent.

It remains to show that the supremum of f^* is finite, *i.e.* that $f^* \in C^0(\overline{\Omega})$. We use CHEBYSHEV's inequality (see Theorem A.5) with

$$a = \|f\|_{L^p(\Omega)} 2^{\frac{1}{p}} |\Omega \cap B_1(x)|^{-\frac{1}{p}} ,$$

which gives

$$\left|\left\{x \in \overline{\Omega} : |f| \geqslant \|f\|_{L^p(\Omega)} 2^{\frac{1}{p}} |\Omega \cap B_1(x)|^{-\frac{1}{p}}\right\}\right| \leqslant \frac{|\Omega \cap B_1(x)|}{2} .$$

In other words, for all $x \in \overline{\Omega}$, we can find a set

$$\Omega_{x,a} \subset \Omega \cap B_1(x) , \quad \text{with} \quad |\Omega_{x,a}| \geqslant \tfrac{1}{2}|\Omega \cap B_1(x)| > 0 ,$$

such that f is bounded on $\Omega_{x,a}$ by a. We now pick $y \in \Omega_{x,a}$. Then, from the previous estimate for the $C^{0,\alpha}$-semi-norm of f^*, we get for every $x \in \overline{\Omega}$

$$|f^*(x)| \leqslant |f^*(x) - f^*(y)| + |f^*(y)| \leqslant c(n,p,\alpha,A)[f]_{\mathcal{L}^{p,n+p\alpha}(\Omega)} + c(p,A)\|f\|_{L^p(\Omega)} . \tag{2.7}$$

All in all, we have shown

$$\|f^*\|_{C^{0,\alpha}(\bar{(\Omega)})} \leqslant c(n,p,\alpha,A)\|f\|_{\mathcal{L}^{p,n+p\alpha}(\Omega)} ,$$

and hence f possesses a representative in $C^{0,\alpha}(\Omega)$ and the proof is complete. □

As a consequence we have

Corollary 2.19. *Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, with LIPSCHITZ boundary. Let $p > n$ and $u \in W^{1,p}(\Omega)$. Then $u \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ and*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\overline{\Omega})} \leqslant c\|u\|_{W^{1,p}(\Omega)} ,$$

with $c = c(\Omega, p)$.

Proof. Pick $x_0 \in \overline{\Omega}$ and $\rho > 0$. We want to estimate $\int_{\Omega\cap B_\rho(x_0)} |u - u_{\Omega\cap B_\rho(x_0)}| \,\mathrm{d}x$. If we work with u directly, we are able to get a factor $\rho^{n-\frac{n}{p}}$ using the POINCARÉ-WIRTINGER and HÖLDER's inequalities, but this is not enough. Extend u to a function $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ with

$$\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leqslant c_1\|u\|_{W^{1,p}(\Omega)} .$$

Using the POINCARÉ-WIRTINGER and the HÖLDER's inequalities,

$$\begin{aligned}
\int_{\Omega\cap B_\rho(x_0)} |u - u_{B_\rho(x_0)}| \,\mathrm{d}x &\leqslant \int_{B_\rho(x_0)} |\tilde{u} - \tilde{u}_{B_\rho(x_0)}| \,\mathrm{d}x \\
&\leqslant c_2\rho \int_{\mathbb{R}^n} |D\tilde{u}| \,\mathrm{d}x \\
&\leqslant c_3 \left(\int_{B_\rho(x_0)} |D\tilde{u}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \rho^{n-\frac{n}{p}+1} ,
\end{aligned}$$

that is, $u \in \mathcal{L}^{1,n+1-\frac{n}{p}}(\Omega) \cong C^{0,1-\frac{n}{p}}(\overline{\Omega})$. □

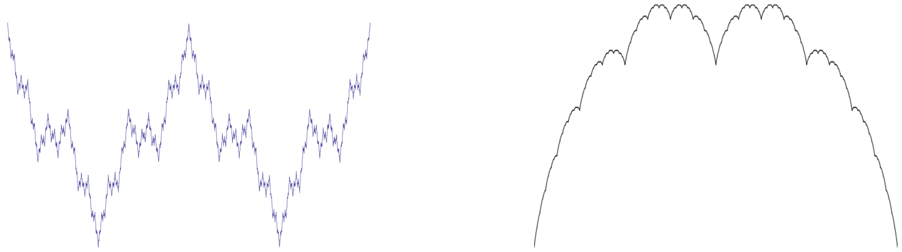

Figure 2: Illustration of a Weierstraß function (left) and a Blancmange curve (right)

From this, we have a generalization of Morrey's inequality, or *Sobolev-Morrey* embedding theorem to cases other than just $p > n$:

Theorem 2.20 (Morrey's theorem on the growth of the Dirichlet integral)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $1 \leqslant p \leqslant n$. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is such that $Du \in L^{p,n-p+p\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$, then $u \in C^{0,\alpha}(\Omega)$.*

Proof. Again using the Poincaré-Wirtinger and the Hölder inequalities, we have for any ball $B_\rho(x_0) \Subset \Omega$

$$\int_{B_\rho(x_0)} |u - u_{B_\rho(x_0)}|^p \,\mathrm{d}x \leqslant c\rho^p \int_{B_\rho(x_0)} |Du|^p \,\mathrm{d}x \leqslant c\rho^{n+p\alpha} \|Du\|^p_{L^{p,n-p+p\alpha}(\Omega)} .$$

Using coverings, we get that $u \in \mathcal{L}^{p,n+p\alpha}_{\mathrm{loc}}(\Omega)$, and we finish the proof by applying Campanato's theorem. □

Remark 2.21. *The converse statement cannot hold, as Hölder continuous functions are not necessarily weak differentiable. Examples are the Weierstraß functions*

$$f(x) := \sum_{k \geqslant 0} a^k \cos(b^k \pi x) , \quad \text{with} \quad 0 < a < 1 \quad \text{and} \quad ab \geqslant 1 ,$$

and the Blancmange curves

$$f(x) := \sum_{k \geqslant 0} \frac{s(2^k x)}{2^k} ,$$

where s is the triangle wave function $s(x) := \min_{z \in \mathbb{Z}} |x - z|$.

Definition 2.22 (Bounded mean oscillation)**.** *Let $Q_0 \subset \mathbb{R}^n$ be an n-dimensional cube. We say that a function $u \in L^1_{\mathrm{loc}}(Q_0)$ is of* bounded mean oscillation*, which we write $u \in \mathrm{BMO}(Q_0)$, if*

$$[u]_{\mathrm{BMO}} := \sup \fint_Q |u - u_{B_Q}| \,\mathrm{d}x < \infty ,$$

where

$$u_Q = \fint_Q u \,\mathrm{d}x$$

is the average of u over Q and the supremum is taken over all n-cubes $Q \subset Q_0$ whose sides are parallel to those of Q_0. Alternatively, one can use balls in the definition.

A fundamental result for BMO functions is exponential integrability:

[3]Fritz John (1910-1994) German-born American mathematician. He worked on the Radon transform, water waves and nonlinear elasticity.

[4]Louis Nirenberg (1925-2020) tasked Nash with the problem of regularity of solutions to parabolic equations with rough coefficients in 1956 during his stay at the Courant Institute. He was awarded the Abel Prize in 2015 together with Nash for his work.

Lemma 2.23 (JOHN[3]-NIRENBERG[4] Inequality [6][5, Theorem 7.21]). *Suppose that* $v \in \mathrm{BMO}(\Omega)$. *Then there are positive constants* c_1 *and* c_2, *depending on* n *and* $[v]_{\mathrm{BMO}}$ *only, such that for every* $B_{2r}(y) \subset \Omega$, *we have*

$$\fint_{B_r(y)} \exp\left(c_1|v - v_{B_r(y)}|\right) \, \mathrm{d}x \leqslant c_2 \,.$$

The proof is a bit involved and —time permitting— will be covered at the end of this lecture.

3 Elliptic equations

3.1 Inner regularity

In this section, we consider the following linear equation, set on some domain (*i.e.* nonempty, open, connected set) of $\mathbb{R}^n$, where $n \geqslant 2$:

$$\operatorname{div}(\mathbb{A}(x)\,\nabla u(x)) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^j}u(x)\right) = 0\,, \tag{3.1}$$

where the matrix $\mathbb{A}(x) = [a^{ij}(x)]_{1\leqslant i,j\leqslant n}$ has measurable entries. If u is such that (3.1) holds, we will write $L_{\mathbb{A}}u = 0$ for short. We will also need the ellipticity and boundedness of $\mathbb{A}$, that is, there exists $\Lambda \geqslant 1$ such that

$$\Lambda^{-1}I \preccurlyeq \mathbb{A}(x) \preccurlyeq \Lambda I$$

for almost all x, where the inequality is meant in the sense of symmetric matrices. In other words, for almost all x and for all $\xi \in \mathbb{R}^n$, we require

$$\Lambda^{-1} \leqslant \langle \xi, \mathbb{A}(x)\,\xi\rangle \leqslant \Lambda\,, \tag{3.2}$$

for some $\Lambda \geqslant 1$. If $\mathbb{A}$ is diagonalizable and $(\lambda_i(x))_{1\leqslant i\leqslant n}$ denote the eigenvalues of $\mathbb{A}(x)$, this condition is equivalent to $\Lambda^{-1} \leqslant \lambda_i(x) \leqslant \Lambda$ for almost every x. This also means that $\mathbb{A}$ is positive definite.

Definition 3.1 (Weak solution)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that $u \in W^{1,2}(\Omega)$ is a* weak solution *of* (3.1) *on Ω if the equation*

$$\int_{\Omega} \langle \nabla\phi(x), \mathbb{A}(x)\,\nabla u(x)\rangle\,\mathrm{d}x = 0$$

holds for all test functions $\phi \in C_c^\infty(\Omega)$.

Theorem 3.2 (De Giorgi, [3], Nash [10], Moser [8, 9])**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (3.1) *where $\mathbb{A}$ fulfils the condition* (3.2)*. Then there exists $0 < \alpha(n,\Lambda) \leqslant 1$ such that $u \in C^{0,\alpha}(\Omega')$ for all $\Omega' \Subset \Omega$. Moreover, there exists $C(n,\Lambda,\Omega,\Omega') > 0$ such that the following norm estimate holds:*

$$\|u\|_{C^{0,\alpha}(\Omega')} \leqslant C\|u\|_{L^2(\Omega)}\,.$$

3.1.1 The method of De Giorgi

We follow Vasseur [14].

Remark 3.3. *Let $u \in W^{1,2}$ be a weak solution of* (3.1) *on some domain Ω, where $\mathbb{A}$ fulfils* (3.2)*. Scalings and translations of u solve a similar equation, for which* (3.2) *still holds. More precisely: let $\lambda > 0$, $x_0 \in \Omega$ and $\varepsilon > 0$ and define*

$$v(y) := \lambda u(x_0 + \varepsilon y)\,,$$

for $y \in \widetilde{\Omega} \subset \{y \in \mathbb{R}^n : x_0 + \varepsilon y \in \Omega\}$. Let $\varphi \in C_c^\infty(\widetilde{\Omega})$ and define $\phi := y \mapsto \varphi(x_0 + \varepsilon y)$, $\mathbb{B} := y \mapsto \mathbb{A}(x_0 + \varepsilon y)$. We then have

$$\begin{aligned}\int_{\widetilde{\Omega}} \langle \nabla_y v(y), \mathbb{B}(y)\,\nabla_y \varphi(y)\rangle\,\mathrm{d}y &= \int_{\widetilde{\Omega}} \langle \varepsilon\lambda\nabla_x u(x_0+\varepsilon y), \mathbb{A}(x_0+\varepsilon y)\,\varepsilon\nabla_x\phi(x_0+\varepsilon y)\rangle\,\mathrm{d}y\,,\\ &= \varepsilon^{-n+2}\lambda\int_{\Omega} \langle \nabla_x u(x), \mathbb{A}(x)\,\nabla_x\phi(x)\rangle\,\mathrm{d}x\\ &= 0\,,\end{aligned}$$

since $\phi \in C_c^\infty(\Omega)$. It follows that v in a weak solution of $L_{\mathbb{B}} = 0$ on $\widetilde{\Omega}$, where the condition (3.2) *holds for $\mathbb{B}$.*

Remark 3.4. *We will prove Theorem 3.2 on the balls B_1 and $B_{1/2}$, this is without loss of generality. By translation and scaling, one can get the result for any balls, as follows. Let $d = \operatorname{dist}(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$, we define*

$$v(x) = u(x_0 + d \cdot x)\,,$$

where $x \in B_1$, so that by Remark 3.3, v is a weak solution $L_{\mathbb{B}} = 0$ on B_1, where $\mathbb{B}$ fulfils (3.2). Then $v \in C^{0,\alpha}(B_{1/2})$ where α depends neither on x_0 nor on d, so that $u \in C^{0,\alpha}(\Omega')$,

The proof of De Giorgi can be split in two steps. First, one derives an estimate on the supremum of u, using the control provided by the so-called energy. Second, one shows, using the estimate in $L^\infty(\Omega')$, that u is in fact in $C^{0,\alpha}(\Omega')$.

First step: the supremum bound

Lemma 3.5. *There exists $\delta^* > 0$, depending on n and Λ only, such that for any $u \in W^{1,2}(B_1)$ weak solution of (3.1) in B_1, where $\mathbb{A}$ fulfils (3.2), we have the following. If*

$$\|u_+\|_{L^2(B_1)} \leqslant \delta^*\,,$$

then

$$\|u_+\|_{L^\infty(B_{1/2})} \leqslant \tfrac{1}{2}\,.$$

Remark 3.6. *Note that the constant δ^* is independent of u, this is key. By scaling, similar results holds for balls of radius different from 1 and 1/2, albeit for a different constant δ^*. This already offers some regularization: if the L^2-norm of u is small enough in a ball, u is actually bounded (a.e.) on a ball with half the radius. However, u may be unbounded near the boundary of the larger ball.*

Before we dive in the proof itself, we introduce some notation, and comment on the general idea.

For $0 \leqslant k \leqslant 1$, we define the family of nested balls

$$B'_k := B_{1/2(1+2^{-k})}\,,$$

which are so that $B'_0 = B_1$ and $B'_k \to B_{1/2}$ as $k \to +\infty$. We also define a corresponding sequence of "energy levels" e_k:

$$e_k := \tfrac{1}{2}\left(1 - 2^{-k}\right),$$

and

$$u_k = (u - e_k)_+$$

as well as

$$U_k = \int_{B'_k} |u_k(x)|^2 \,\mathrm{d}x\,.$$

Note that $U_0 = \|u_+\|^2_{L^2(B_1)}$.

We want to derive an estimate of the form

$$U_{k+1} \leqslant C^k U_k^\beta\,, \tag{3.3}$$

where $C > 1$ and $\beta > 1$. We stress that this inequality is *nonlinear*, even *superlinear*, which is crucial: On the one hand, for the first factor, we have that $\lim_{k\to\infty} C^k = +\infty$. On the other hand, $\beta > 1$, this means that if U_0 (the L^2-norm of u_+ on B_1) is small enough, the nonlinear factor will "beat" the factor C^k, and the sequence U_k can be shown to converge to 0. This means that at the limit we have

$$\int_{B_{1/2}} \left(u(x) - \tfrac{1}{2}\right)_+^2 \,\mathrm{d}x = 0\,,$$

so that $u(x) \leqslant \frac{1}{2}$ almost everywhere in $B_{1/2}$, *i.e.* $u \in L^\infty(B_{1/2})$.

The nonlinear inequality (3.3) is obtained using elementary results. Two of them are essentially linear in nature, the SOBOLEV's imbedding theorem, and the energy estimate, see Lemma 3.7 below. The third is nonlinear however, MARKOV's inequality, which states that for a positive, measurable function f, we have

$$|\{x : f(x) \geqslant a\}| \leqslant \frac{\|f\|_{L^1}}{a}\,,$$

for any $a > 0$. See Appendix A for a proof. Note that this result can be generalized by replacing the right-hand side by $\|f\|_{L^p}^p / a^p$, becoming CHEBYSHEV's inequality.

In order to use the nonlinear estimate, we need to pay in terms of the level sets of f, which is why we introduced the level sets e_k.

The proof is split in three steps:

Step 1: The energy estimate We first start by proving the following:

Lemma 3.7 (Energy estimate)**.** *Let $u \in W^{1,2}(B_r)$ be a weak solution of $L_{\mathbb{A}} u = 0$, where $\mathbb{A}$ fulfils (3.2), and let $\phi \in C_c^\infty(B_r)$. Then there exists $C > 0$ independent of u such that the following inequality holds:*

$$\int_{B_r} |\nabla(\phi u_+)|^2 \,\mathrm{d}x \leqslant C\|\nabla\phi\|_{L^\infty}^2 \int_{B_r \cap \operatorname{supp}\phi} u_+^2 \,\mathrm{d}x\,.$$

Moreover, $C = \Lambda^2$ if $\mathbb{A}$ is symmetric.

In the literature, such estimates, which offer the control of some L^2-norm of ∇u in terms of some L^2-norm of u are known as *CACCIOPPOLI (type) inequalities*. The idea at the core of the proof of DE GIORGI is to this estimate if iterated on smaller and smaller balls. A similar iteration procedure was used by NASH and MOSER in their proofs. Theorem 3.2 can also be proven in a continuous way, by integrating instead of iterating, see [4, Section 8.5] and [13] for the original paper.

Proof. We test the equation with $\phi^2 u_+$ to get

$$\int_{B_r} \langle \nabla(\phi^2 u_+), \mathbb{A}\,\nabla u_+\rangle \,\mathrm{d}x = 0\,.$$

Since we are interested in the L^2-norm of $\nabla(\phi u_+)$, we somehow need to symmetrize this expression and move one ϕ from the left to the right. We have

$$\begin{aligned}
0 &= \int_{B_r} \langle \nabla(\phi^2 u_+), \mathbb{A}\nabla u_+\rangle \,\mathrm{d}x \\
&= \int_{B_r} \langle \phi\nabla(\phi u_+), \mathbb{A}\,\nabla u_+\rangle \,\mathrm{d}x + \int_{B_r} \langle (\phi u_+)\nabla\phi, \mathbb{A}\,\nabla u_+\rangle \,\mathrm{d}x \\
&= \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\,\nabla(\phi u_+)\rangle \,\mathrm{d}x - \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\, u_+\nabla\phi\rangle \,\mathrm{d}x + \int_{B_r} \langle (\phi u_+)\nabla\phi, \mathbb{A}\,\nabla u_+\rangle \,\mathrm{d}x \\
&= \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\,\nabla(\phi u_+)\rangle \,\mathrm{d}x - \int_{B_r} \langle \nabla(\phi u_+), (\mathbb{A} - \mathbb{A}^T)\, u_+\nabla\phi\rangle \,\mathrm{d}x \\
&\quad - \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}^T\, u_+\nabla\phi\rangle \,\mathrm{d}x + \int_{B_r} \langle (\phi u_+)\nabla\phi, \mathbb{A}\,\nabla u_+\rangle \,\mathrm{d}x \\
&= \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\,\nabla(\phi u_+)\rangle \,\mathrm{d}x - \int_{B_r} \langle \nabla(\phi u_+), (\mathbb{A} - \mathbb{A}^T)\, u_+\nabla\phi\rangle \,\mathrm{d}x \\
&\quad - \int_{B_r} \langle u_+^2\nabla\phi, \mathbb{A}^T\,\nabla\phi\rangle \,\mathrm{d}x - \int_{B_r} \underline{\langle (\phi u_+)\nabla u_+, \mathbb{A}^T\,\nabla\phi\rangle} \,\mathrm{d}x + \int_{B_r} \underline{\langle (\phi u_+)\nabla\phi, \mathbb{A}\,\nabla u_+\rangle} \,\mathrm{d}x\,.
\end{aligned}$$

- If $\mathbb{A}$ is symmetric, the second term vanishes and we get:

$$\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\,\nabla(\phi u_+)\rangle \,\mathrm{d}x = \int_{B_r} u_+^2\langle \nabla\phi, \mathbb{A}^T\,\nabla\phi\rangle \,\mathrm{d}x\,,$$

and at this point, the ellipticity condition (3.2) yields

$$\int_{B_r} |\nabla(\phi u_+)|^2 \,\mathrm{d}x \leqslant \Lambda^2 \int_{B_r} u_+^2 |\nabla\phi|^2 \,\mathrm{d}x \leqslant \Lambda^2 \|\nabla\phi\|_{L^\infty}^2 \int_{B_r \cap \operatorname{supp}\phi} u_+^2 \,\mathrm{d}x \,.$$

- If $\mathbb{A}$ is not symmetric, we estimate the second term as follows:

$$\begin{aligned}
\left|\int_{B_r} \langle \nabla(\phi u_+), (\mathbb{A}-\mathbb{A}^T) u_+ \nabla\phi\rangle \,\mathrm{d}x\right| &\leqslant \int_{B_r} |\langle \nabla(\phi u_+), \mathbb{A} u_+ \nabla\phi\rangle| + |\langle \mathbb{A}\nabla(\phi u_+), u_+\nabla\phi\rangle| \,\mathrm{d}x \\
&\overset{\text{cont.}}{\leqslant} 2\Lambda \|\nabla(\phi u_+)\|_{L^2} \|u_+\nabla\phi\|_{L^2} \\
&\overset{\text{ellipt.}}{\leqslant} 2\Lambda^{\frac{3}{2}} \left(\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+)\rangle \,\mathrm{d}x\right)^{\frac{1}{2}} \|u_+\nabla\phi\|_{L^2} \\
&\overset{\text{Young}}{\leqslant} \frac{1}{2}\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+)\rangle \,\mathrm{d}x + 2\Lambda^3 \int_{B_r} u_+^2 |\nabla\phi|^2 \,\mathrm{d}x
\end{aligned}$$

After plugin this back into the expression above, this gives

$$0 \geqslant \frac{1}{2}\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+)\rangle \,\mathrm{d}x - \Lambda(1+2\Lambda^2) \int_{B_r} u_+^2 |\nabla\phi|^2 \,\mathrm{d}x \,.$$

and after making use of the ellipticity:

$$\int_{B_r} |\nabla(\phi u_+)|^2 \,\mathrm{d}x \leqslant 2\Lambda^2(1+2\Lambda^2) \int_{B_r} u_+^2 \,|\nabla\phi|^2 \,\mathrm{d}x \leqslant 2\Lambda^2 \left(1+\Lambda^2\right) \|\nabla\phi\|_{L^\infty}^2 \int_{B_r \cap \operatorname{supp}\phi} u_+^2 \,\mathrm{d}x \,,$$

which is the estimate we wanted.

□

Step 2 We will now use our freshly derived energy estimate (or Caccioppoli inequality) in the nested ball B_k'. To this end, we introduce a family of cut-off functions ϕ_k such that:

$$\phi_k \in C_c^\infty(B_{k-1}') \quad \text{and} \quad \phi_k = 1 \text{ in } B_k' \,,$$

with

$$\|\nabla\phi_k\|_{L^\infty} \leqslant C\, 2^k \,.$$

Note that we have

$$U_k = \int_{B_k'} |u_k|^2 \,\mathrm{d}x \leqslant \int_{B_1} \phi_k^2 u_k^2 \,\mathrm{d}x \,.$$

At this point, let us recall

Theorem 3.8 (Sobolev's inequality [12])**.** *Assume $n \geqslant 3$. We write $2^\star = \frac{2n}{n-2}$. For any smooth, bounded domain $\Omega \subset \mathbb{R}^n$ there exist a constant S such that for any $u \in H_0^1(\Omega)$ we have the following inequality:*

$$\|u\|_{L^{2^\star}(\Omega)}^2 \leqslant S \|\nabla u\|_{L^2(\Omega)}^2 \,. \tag{3.4}$$

See for example [2] for a proof.

We have also $\mathbb{1}_{B_{k+1}'} \leqslant \phi_k \leqslant \mathbb{1}_{B_k'}$ and $u_{k+1} \leqslant u_k$, so that applying Sobolev's inequality to $v = \phi_{k+1} u_{k+1}$ on B_k' we get

$$\left(\int_{B_k'} (\phi_{k+1} u_{k+1})^{2^\star} \,\mathrm{d}x\right)^{2/2^\star} \leqslant S \int_{B_k'} |\nabla(\phi_{k+1} u_{k+1})|^2 \,\mathrm{d}x \,,$$

and using Lemma 3.7 gives

$$\begin{aligned} &\leqslant C2^{2k} \int_{B'_k} |u_{k+1}|^2 \, \mathrm{d}x \\ &\leqslant C2^{2k} \int_{B'_k} |u_k|^2 \, \mathrm{d}x \\ &\leqslant C^k U_k \,. \end{aligned}$$

We can now use MARKOV's inequality:

$$U_{k+1} \leqslant \int_{B'_k} \phi_{k+1}^2 u_{k+1}^2 \, \mathrm{d}x$$

and HÖLDER's inequality with exponents $(n/(n-2), 2/n)$:

$$\leqslant \left(\int_{B'_k} (\phi_{k+1} u_{k+1})^{2^\star} \, \mathrm{d}x \right)^{2/2^\star} |\{\phi_{k+1} u_{k+1} > 0\}|^{2/n}$$

assuming $k \geqslant 2$ we have

$$\begin{aligned} &\leqslant C^k U_k |\{\phi_k u_k > 2^{-(k+2)}\}|^{2/n} = C^k U_k \left| \left\{ (\phi_k u_k)^2 > 2^{-2(k+2)} \right\} \right|^{2/n} \\ &\leqslant \frac{C^k}{2^{-4(k+2)/n}} U_k^{1+2/n} \leqslant 2^{8/n} \left(2^{4/n} C \right)^k U_k^{1+2/n} \,. \end{aligned}$$

This is exactly (3.3) with $\beta = 1 + 2/n$.

Step 3 This is the final step. Recall that we are looking for δ^* such that if $\|u_+\|_{L^2(B_1)} = U_0 \leqslant \delta^*$, then $\|u\|_{L^\infty(B_{1/2})} \leqslant 1/2$. Using a comparison argument, we will now show that U_k converges to zero. Let us define the statement $P(k)$ as

$$P(k) : C^k U_k^{\beta-1} \leqslant \frac{1}{(2C)^{\frac{1}{\beta-1}}} \,. \tag{3.5}$$

We start by showing that $P(k)$ holds for any k, provided that U_0 is small enough. We first choose k_0 such that

$$\frac{1}{2^{k_0}} \leqslant \frac{1}{(2C)^{\frac{1}{\beta-1}}} \,,$$

holds. $U_{k+1} \leqslant U_k$ by definition, so that we can choose δ^* small enough such that for any $U_0 = \|u\|^2_{L^2(B_1)} \leqslant (\delta^*)^2$, $P(k)$ is true for any $k \leqslant k_0$. We now show by induction that $P(k)$ is also true for $k \geqslant k_0$. Fix $k \leqslant k_0$ and assume that $P(i)$ is true for all $i \leqslant k$. Using (3.3) we have

$$U_{k+1} \leqslant \frac{1}{(2C)^{\frac{k+1}{\beta-1}}} \,,$$

so that

$$C^{k+1} U_{k+1}^{\beta-1} \leqslant \frac{1}{2^{k+1}} \leqslant \frac{1}{2^{k_0}} \leqslant \frac{1}{(2C)^{\frac{1}{\beta-1}}} \,,$$

so that $P(k+1)$ is true. It follows that

$$\int_{B_{1/2}} \left(u - \tfrac{1}{2}\right)_+^2 \, \mathrm{d}x = \lim_{k\to\infty} U_k = 0,$$

which implies $\|u_+\|_{L^\infty(B_{1/2})} \leqslant \frac{1}{2}$. This finishes the proof of Lemma 3.5. □

As a consequence of the scaling method mentioned in Remark 3.6, we have the following result:

Corollary 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $u \in W^{1,2}(\Omega)$ be a weak solution of (3.1) in Ω, where $\mathbb{A}$ fulfils (3.2). Then, $u \in L^\infty(\Omega')$ for any $\Omega' \Subset \Omega$.*

Proof. Let $d := \operatorname{dist}(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$, define v on B_1 as

$$v(y) := \delta^* \frac{d^{\frac{n}{2}}}{\|u\|_{L^2(\Omega)}} u(x_0 + dy)\,,$$

where δ^* is the constant given by Lemma 3.5. v is a weak solution of $L_{\mathbb{B}} = 0$ on B_1 for some $\mathbb{B}$ fulfilling condition (3.2) (see Remark 3.3). Moreover, we have $\|v\|_{L^2(B_1)} \leqslant \delta^*$, so that we get $v(y) \leqslant 1/2$ almost everywhere on B_1. Applying the same reasoning for $-v$, we get

$$|v(y)| \leqslant \frac{1}{2} \quad \text{a.e.,}$$

from which it follows that

$$\|u\|_{L^\infty(\Omega')} \leqslant (\delta^*)^{-1} d^{-n/2} \|u\|_{L^2(\Omega)}\,.$$

□

Second step: the Oscillation lemma In this section we deal with the second and last step in the proof of Theorem 3.2, which is the proof of so-called Oscillation Lemma below.

We first start by definition the oscillation:

Definition 3.10 (Oscillation)**.** *For any open set A and any real-valued function f on A, the* oscillation *of f on A is defined as*

$$\operatorname{osc}_A f = \sup_A f - \inf_A f\,.$$

We can now state

Lemma 3.11 (Oscillation Lemma)**.** *Let $u \in W^{1,2}(B_2)$ be a weak solution of (3.1) on B_2 where $\mathbb{A}$ fulfils (3.2). Then there exists a constant $\lambda(\Lambda, n) < 1$ such that*

$$\operatorname{osc}_{B_{1/2}} u \leqslant \lambda \operatorname{osc}_{B_2} u$$

holds.

The De Giorgi-Nash-Moser Theorem follows a consequence of the Oscillation Lemma:

Proof of Theorem 3.2. Take $x_0 \in \Omega' \Subset \Omega$ and let $d := \operatorname{dist}(\Omega', \partial\Omega)$. We introduce the family of rescaled functions v_k, defined on B_2:

$$\begin{aligned} v_1(y) &= u\left(x_0 + \tfrac{d}{2}\, y\right)\,, \\ v_k(y) &= v_{k-1}(y/4) = u\left(x_0 + \tfrac{1}{4^{k-1}} \tfrac{d}{2}\, y\right)\,, \end{aligned}$$

As we have already seen, the functions v_k are weak solutions of $L_{\mathbb{B}_k} v_k = 0$ where

$$\mathbb{B}_k(y) = \mathbb{A}\left(x_0 + \tfrac{1}{4^{k-1}} \tfrac{d}{2}\, y\right)$$

again fulfils (3.2), with the same constant Λ. With this, we can apply Lemma 3.11 recursively on the v_k, which gives

$$\begin{aligned} \operatorname{osc}_{B_{1/2}} v_{k+1} &\leqslant \lambda \operatorname{osc}_{B_2} v_{k+1} \\ &\leqslant \lambda \operatorname{osc}_{B_2} (y \mapsto v_k(y/4)) \\ &\leqslant \lambda \operatorname{osc}_{B_{1/2}} v_k \\ &\leqslant \lambda^k \operatorname{osc}_{B_{1/2}} v_1 \\ &\leqslant 2\lambda^k \|u\|_{L^\infty(\Omega')} \end{aligned}$$

Then, we have the following estimate:

$$\sup_{|x_0-x|\leqslant 4^{-(k+1)}d} |u(x_0)-u(x)| \leqslant \operatorname{osc}_{B_{1/2}} v_{k+1} \leqslant 2\lambda^k \|u\|_{L^\infty(\Omega)}\,,$$

which depends neither on d nor x_0. We define the interval $I_k := [4^{-(k+1)}\,d, 4^{-k}\,d]$ and then write

$$\sup_{|x_0-x|\leqslant d} \frac{|u(x_0)-u(x)|}{|x_0-x|^\alpha} = \sup_{k\in\mathbb{N}} \sup_{|x_0-x|\in I_k} \frac{|u(x_0)-u(x)|}{|x_0-x|^\alpha} \leqslant \sup_{k\in\mathbb{N}} \frac{2\lambda^{k-1}\|u\|_{L^\infty(\Omega)}}{4^{-(k+1)\alpha}} = \sup_{k\in\mathbb{N}} \frac{2\lambda^{-1}\|u\|_{L^\infty(\Omega)}}{4^{-\alpha}}(\lambda 4^\alpha)^k\,.$$

Picking

$$\alpha = -\frac{\ln\lambda}{2\lambda 2}\,,$$

the limit of the right-hand side is finite and $u \in C^{0,\alpha}(\Omega')$. □

We can reformulate Lemma 3.11 slightly, in the following way:

Proposition 3.12. *Let $v \leqslant 1$ be a weak solution of $L_{\mathbb{A}}v = 0$ on B_2, with $\mathbb{A}$ fulfiling (3.2). If there exists $\mu > 0$ such that $|B_1 \cap \{v \leqslant 0\}| \geqslant \mu$, then there exists a constant λ depending only on n, μ and Λ such that the following estimate holds:*

$$\sup_{B_{1/2}} v \leqslant 1-\lambda\,.$$

In other words, if v is a solution of $L_{\mathbb{A}}v = 0$ which is smaller than or equal to one on B_2 and which remains "far from one" (nonpositive) on a set of positive measure, v cannot get arbitrarily close to one on $B_{1/2}$.

Let us show how this leads to Lemma 3.11.

Proof of Lemma 3.11. We rescale u between -1 and 1 by considering the function v defined as

$$v(x) := \frac{2}{\operatorname{osc}_{B_2} u}\left(u(x) - \frac{\sup_{B_2} u + \inf_{B_2} u}{2}\right)\,.$$

We have $|v| \leqslant 1$. Assume that $v \leqslant 0$ on (at least) the half of B_1. Applying Proposition 3.12 to v yields that

$$\operatorname{osc}_{B_{1/2}} v = \sup_{B_{1/2}} v - \inf_{B_{1/2}} v \leqslant 1-\lambda-(-1) = 2-\lambda\,,$$

from which it follows from the definition of v that

$$\operatorname{osc}_{B_{1/2}} u = \frac{\operatorname{osc}_{B_2} u}{2} \operatorname{osc}_{B_{1/2}} v \leqslant (1-\lambda/2)\operatorname{osc}_{B_2} u\,.$$

Working with $(-v)$, we get the same result if $v \geqslant 0$ on (at least) the half of B_1, since $\operatorname{osc}_A(-v) = \operatorname{osc}_A v$. □

To prove Proposition 3.12, we may first note that if

$$|B_1 \cap \{v \leqslant 0\}| \geqslant |B_1| - (\delta^*)^2\,,$$

where δ^* is given by Lemma 3.5. then, if follows from the bounds on v that

$$\|v_+\|_{L^2(B_1)} \leqslant \delta^*$$

and Lemma 3.5 imply that $v_+|_{B_{1/2}} \leqslant \frac{1}{2}$.

The main tool is the following inequality of De Giorgi. It may be considered as a quantitative version of the fact that a function with a jump discontinuity cannot be in $W^{1,2}$.

We first need to introduce some more notation. For some measurable function w defined on B_1, we define the following subsets of B_1:

$$\begin{aligned} {}^{0}S_w &:= B_1 \cap \{w \leqslant 0\}\,,\\ {}^{1/2}_{0}S_w &:= B_1 \cap \{0 < w < 1/2\}\,,\\ {}_{1/2}S_w &:= B_1 \cap \{1/2 \leqslant w\}\,. \end{aligned}$$

Lemma 3.13 (De Giorgi's isoperimetric inequality)**.** *There exists a constant $C_n > 0$, depending on n only, such that the following holds: If $w \in W^{1,2}(B_1)$ is such that*

$$\int_{B_1} |\nabla w_+|^2 \,\mathrm{d}x \leqslant C_0 ,$$

then we have

$$C_n \left(|{}_{1/2}S_w| |{}^0S_w|^{1-\frac{1}{n}} \right)^2 \leqslant C_0 |{}_0^{1/2}S_w| .$$

Proof. Consider $\overline{w} := \sup(0, \inf(w, ½))$. Note that the weak derivative of $\overline{w}$ fulfils

$$\nabla \overline{w} = \mathbb{1}_{\{0 \leqslant w \leqslant ½\}} \nabla w_+ .$$

For any $x \in {}^0S_w$, $y \in {}_{1/2}S_w$, we have

$$\begin{aligned} \tfrac{1}{2} \leqslant \overline{w}(y) - \overline{w}(x) &= \int_0^1 \frac{d}{dt} \overline{w}(x + t(y-x)) \,\mathrm{d}t \\ &= \int_0^1 (y-x) \cdot \nabla \overline{w}(x + t(y-x)) \,\mathrm{d}t \end{aligned}$$

set $s = t|y - x|$

$$\leqslant \int_0^{|x-y|} |\nabla \overline{w}(x + s e_\sigma)| \,\mathrm{d}s ,$$

where $e_\sigma = (y-x)/|y-x|$. The integrand in the last integral is nonnegative, so it is increasing in its upper bound. This means that, by "extending" $\nabla \overline{w}$ by 0 outside B_1, we can bound it from above by integrating up to infinity and get

$$\frac{1}{2} \leqslant \int_0^\infty |\nabla \overline{w}(x + s e_\sigma)| \,\mathrm{d}s .$$

We can now integrate with respect to $y \in {}_{1/2}S_w$ to get

$$\begin{aligned} |{}_{1/2}S_w|/2 &\leqslant \int_{{}_{1/2}S_w} \left(\int_0^\infty |\nabla \overline{w}(x + s e_\sigma)| \,\mathrm{d}s \right) \mathrm{d}y \\ &\leqslant \int_{B_1} \left(\int_0^\infty |\nabla \overline{w}(x + s e_\sigma)| \,\mathrm{d}s \right) \mathrm{d}y \end{aligned}$$

We change to polar coordinates

$$\begin{aligned} &\leqslant \int_0^1 r^{n-1} \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty |\nabla \overline{w}(x + s e_\sigma)| \,\mathrm{d}s \right) \mathrm{d}\sigma \,\mathrm{d}r \\ &\leqslant \int_{\mathbb{S}^{n-1}} \int_0^\infty |\nabla \overline{w}(x + s e_\sigma)| \,\mathrm{d}s \,\mathrm{d}\sigma \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty s^{n-1} \frac{|\nabla \overline{w}(x + s e_\sigma)|}{s^{n-1}} \,\mathrm{d}s \,\mathrm{d}\sigma \\ &= \int_{B_1} \frac{|\nabla \overline{w}(y)|}{|x-y|^{n-1}} \,\mathrm{d}y , \end{aligned}$$

where the last equality holds with our abuse of notation. We can now integrate with respect to $x \in {}^0S_w$ to find

$$|{}^0S_w| |{}_{1/2}S_w|/2 \leqslant \int_{B_1} |\nabla \overline{w}(y)| \underbrace{\left(\int_{{}^0S_w} \frac{\mathrm{d}x}{|y-x|^{n-1}} \right)}_{=: I_{{}^0S_w}(y)} \mathrm{d}y .$$

Since the integrand in $I_{{}^0S_w}$ is a positive nonincreasing function of $|y-x|$, $I_{{}^0S_w}$ is maximized (among the sets of measure $|{}^0S_w|$) by the ball of radius $(n|{}^0S_w|/|\mathbb{S}^{n-1}|)^{1/n}$ centred on y. In other words we have the following inequality, for any y,

$$I_{{}^0S_w}(y) \leqslant \int_{\mathbb{S}^{n-1}} \mathrm{d}\sigma \int_0^{(n|{}^0S_w|/|\mathbb{S}^{n-1}|)^{1/n}} r^{n-1}\frac{\mathrm{d}r}{r^{n-1}} \leqslant c_n|{}^0S_w|^{1/n}\,,$$

where $|\mathbb{S}_{n-1}|$ denotes the surface area of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$. Outside ${}^{1/2}_{0}S_w$, the integrand in the integral over B_1 is zero, so, after making use of the Cauchy-Schwarz inequality, we get

$$|{}^0S_w||{}_{1/2}S_w|/2 \leqslant c_n|{}^0S_w|^{1/n}\left(\int_{{}^{1/2}_{0}S_w}|\nabla w_+|^2\,\mathrm{d}x\right)^{\frac12}|{}^{1/2}_{0}S_w|^{\frac12}\,.$$

Since $\int_{{}^{1/2}_{0}S_w}|\nabla w_+|^2\,\mathrm{d}x \leqslant C_0$, the proof is complete. □

Proof of Proposition 3.12. We consider the new sequence of truncations

$$w_k := 2^k(v-(1-2^{-k})) = 2^k(v-1)+1\,.$$

Note that for any k, we have that $w_k \leqslant 1$ and $w_{k+1} = 2w_k - 1$. Making use of the energy estimate (Lemma 3.7) with $r=2$ and $\mathbb{1}_{B_1}\leqslant\phi\leqslant\mathbb{1}_{B_2}$, we have

$$\int_{B_1}|\nabla(w_k)_+|^2\,\mathrm{d}x \leqslant \int_{B_2}|\nabla(\phi w_k)_+|^2\,\mathrm{d}x \leqslant C\int_{B_1}|(w_k)_+|^2\,\mathrm{d}x \leqslant C_0\,.$$

We also have $|B_1\cap\{w_k\leqslant 0\}|\geqslant\mu$. It is now possible to apply Lemma 3.13 on w_k iteratively, as long as

$$\int_{B_1}(w_{k+1})_+^2\,\mathrm{d}x \geqslant (\delta^*)^2\,. \tag{3.6}$$

Assume that (3.6) holds for some k, so that

$$|B_1\cap\{w_{k+1}\geqslant 0\}| = \underbrace{|B_1\cap\{2w_k\geqslant 1\}|}_{{}_{1/2}S_{w_k}} \geqslant \int_{B_1}(w_{k+1})_+^2\,\mathrm{d}x \geqslant (\delta^*)^2\,.$$

From Lemma 3.13, there exists a constant $C_n>0$ which does not depend on k such that

$$|B_1\cap\{0<w_k<½\}| \geqslant C_n/C_0(|{}_{1/2}S_{w_k}||{}^0S_{w_k}|^{1-1/n})^2\,.$$

Note also that by assumption, we have

$$|{}^0S_{w_k}| = |B_1\cap\{w_k\leqslant 0\}| \geqslant |{}^0S_{w_{k-1}}| \geqslant |{}^0S_{w_0}| = |B_1\cap\{v\leqslant 0\}| \geqslant \mu > 0\,.$$

Putting is all together, there is a constant $\gamma>0$ such that

$$|B_1\cap\{0<w_k<½\}| \geqslant \gamma\,.$$

Then,

$$|{}^0S_{w_k}| \geqslant |{}^0S_{w_{k-1}}| + \gamma \geqslant \mu + k\gamma\,,$$

which fails for k large, say for $k\geqslant k_0$. Then,

$$\int_{B_1}(w_{k_0+1})_+^2\,\mathrm{d}x \leqslant (\delta^*)^2\,.$$

Lemma 3.5 then implies that $w_{k_0+1}\leqslant\frac12$ in $B_{½}$. Rescaling back to v gives the result with $\lambda = 2^{-(k_0+2)}$. □

3.1.2 The iteration of Moser

We follow Zhong [16].

In this section, we will present the alternative approach of Moser, which he published in [9]. The main goal is to derive the Harnack type inequality which follows.

But first, we let us extend our notion of solution to subsolutions:

Definition 3.14 (Sub- and supersolutions). *We say that $u \in W^{1,2}(\Omega)$ is a* weak subsolution *of* (3.1) *if*

$$\int_\Omega \langle \nabla \phi(x), \mathbb{A}(x)\nabla u(x)\rangle \,\mathrm{d}x \leqslant 0$$

holds for all $\phi_c^\infty(\Omega)$. We define weak supersolutions *similarly, by changing the direction of the inequality.*

We can now state the following:

Theorem 3.15 (Harnack's inequality). *Let $u \in W^{1,2}(\Omega)$ be a* nonnegative, *weak subsolution of* (3.1), *where $\mathbb{A}$ is symmetric and satisfies* (3.2). *Then, there is a constant $c(n,\Lambda) > 0$ such that for every ball $B_r(y) \subset \Omega$ we have*

$$\sup_{B_{r/2}(y)} u \leqslant c \inf_{B_{r/2}(y)} u\,.$$

Remark 3.16. *Note that, here, $\mathbb{A}$ is assumed to be* symmetric, *as opposed to Section 3.1.1.*

As a consequence, we have:

Theorem 3.17. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (3.1) *where $\mathbb{A}$ is symmetric and fulfils* (3.2). *Then there is $0 < \alpha(n,\Lambda) \leqslant 1$, such that $u \in C^{0,\alpha}(\Omega)$. Moreover, for every ball $B_R(y) \subset \Omega$ and all $0 < r \leqslant R < \infty$, we have*

$$\operatorname{osc}_{B_r(y)} u \leqslant 2^\alpha \left(\frac{r}{R}\right)^\alpha \operatorname{osc}_{B_R(y)} u\,.$$

The proof is again divided in two parts, for the *sup* part and the *inf* part, respectively.

Harnack's inequality: sup Let us first prove the local boundedness of weak solutions

Lemma 3.18. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (3.1) *where $\mathbb{A}$ is symmetric and satisfies* (3.2). *Then $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, for every ball $B_r(y) \subset \Omega$, we have*

$$\sup_{B_{r/2}(y)} |u| \leqslant c\left(\fint_{B_r(y)} |u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}},$$

where $c = c(n,\Lambda) > 0$.

As in the previous section, we will proceed by iteration. The core tools are a Caccioppoli type inequality, and, again, Sobolev's inequality. Let us start with the following inequality (compare with Lemma 3.7).

Lemma 3.19. *Let $u \in W^{1,2}(\Omega)$ be weak solution of* (3.1) *where $\mathbb{A}$ is symmetric and fulfils* (3.2). *Then, for any $\alpha \geqslant 0$ such that $u \in L^{\alpha+2}_{\mathrm{loc}}(\Omega)$ and any $\eta \in C_c^\infty(\Omega)$, we have*

$$\int_\Omega |u|^\alpha |\nabla u|^2 \eta^2 \,\mathrm{d}x \leqslant c\int_\Omega |u|^{\alpha+2}|\nabla\eta|^2 \,\mathrm{d}x\,,$$

where $c = c(\Lambda) > 0$.

Proof. Fix $\eta \in C_c^\infty(\Omega)$ and let $t \geqslant 0$. Define $v := (u-t)_+$. We test the equation (3.1) with $\phi = v\eta^2 \in H_0^1(\Omega)$ and obtain

$$0 = \int_\Omega \langle \nabla\phi, \mathbb{A}\nabla u\rangle \,\mathrm{d}x = \int_\Omega \langle \nabla(u-t)_+, \mathbb{A}\nabla u\rangle \eta^2 \,\mathrm{d}x + 2\int_\Omega \langle \nabla\eta, \mathbb{A}\nabla u\rangle (u-t)_+\eta \,\mathrm{d}x\,. \tag{3.7}$$

To estimate the last integral, we use the CAUCHY-SCHWARZ inequality:

$$|\langle \nabla\eta, \mathbb{A}\nabla u\rangle| \leqslant \langle \nabla u, \mathbb{A}\nabla u\rangle^{\frac{1}{2}} \langle \nabla\eta, \mathbb{A}\nabla\eta\rangle^{\frac{1}{2}}$$

and Hölder's inequality. After squaring, this gives

$$\int_{\{u>t\}} \langle \nabla u, \mathbb{A}\nabla u\rangle \eta^2 \,\mathrm{d}x \leqslant 4\int_{\{u>t\}} \langle \nabla\eta, \mathbb{A}\nabla\eta\rangle |(u-t)_+|^2 \,\mathrm{d}x\,,$$

At this point, we use the boundedness and ellipticity condition (3.2) to get

$$\int_{\{u>t\}} |\nabla u|^2\eta^2 \,\mathrm{d}x \leqslant 4\Lambda^2 \int_{\{u>t\}} |(u-t)_+|^2 |\nabla\eta|^2 \,\mathrm{d}x \leqslant 4\Lambda^2 \int_{\{u>t\}} |u_+|^2 |\nabla\eta|^2 \,\mathrm{d}x\,.$$

Now, the above inequality holds for all $t \geqslant 0$. Multiplying both sides by $\alpha t^{\alpha-1}$ and integrating with respect to t over $(0,\infty)$ we get

$$\int_0^\infty \alpha t^{\alpha-1}\left(\int_{\{u>t\}} |\nabla u|^2\eta^2 \,\mathrm{d}x\right) \mathrm{d}t \leqslant 4\Lambda^2 \int_0^\infty \alpha t^{\alpha-1}\left(\int_{\{u>t\}} |u_+|^2|\nabla\eta|^2 \,\mathrm{d}x\right) \mathrm{d}t\,,$$

and using Fubini-Tonelli:

$$\int_\Omega |\nabla u_+|^2\eta^2 \left(\int_0^\infty \alpha t^{\alpha-1}\mathbb{1}_{\{u>t\}}(x)\,\mathrm{d}t\right) \mathrm{d}x \leqslant 4\Lambda^2 \int_\Omega |u_+|^2|\nabla\eta|^2 \left(\int_0^\infty \alpha t^{\alpha-1}\mathbb{1}_{\{u>t\}}(x)\,\mathrm{d}t\right) \mathrm{d}x\,.$$

Note now that $\{u > t\} = \{u_+ > t\}$, so we can replace the upper bound in the inner integral by u_+ and get:

$$\int_\Omega |u|^\alpha |\nabla u_+|^2\eta^2 \,\mathrm{d}x \leqslant 4\Lambda^2 \int_\Omega |u_+|^{\alpha+2}|\nabla\eta|^2 \,\mathrm{d}x\,.$$

Similarly, we get the same estimate for u_-, and sum the two to get the result. □

SOBOLEV's inequality now gives

Lemma 3.20. *Assume $n \geqslant 3$ and recall that the critical SOBOLEV exponent is given by $2^\star = 2n/(n-2)$. Let $u \in W^{1,2}(\Omega)$ be a weak solution of (3.1) where $\mathbb{A}$ symmetric satisfies (3.2). Then, for any $\alpha \geqslant 0$, $u \in L^{(\alpha+2)2^\star/2}_{\mathrm{loc}}(\Omega)$ if $u \in L^{\alpha+2}_{\mathrm{loc}}(\Omega)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$,*

$$\left(\int_\Omega |u|^{\frac{2^\star(\alpha+2)}{2}}\eta^{2^\star} \,\mathrm{d}x\right)^{\frac{2}{2^\star}} \leqslant c(\alpha+2)^2 \int_\Omega |u|^{\alpha+2}|\nabla\eta|^2 \,\mathrm{d}x\,, \tag{3.8}$$

where $c = c(n,\Lambda) > 0$.

Proof. Define $v := |u|^{\frac{\alpha}{2}}u\eta$, so that

$$\nabla v = \left(\tfrac{\alpha}{2}+1\right)|u|^{\frac{\alpha}{2}}\eta\nabla u + |u|^{\frac{\alpha}{2}}u\nabla\eta\,.$$

To estimate the L^2-norm of ∇v, we use Young's inequality and Lemma 3.19 to get

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x \leqslant c(\alpha+2)^2 \int_\Omega |u|^{\alpha+2}|\nabla\eta|^2 \,\mathrm{d}x\,,$$

We can use Sobolev's inequality for v since $n \geqslant 3$, which then yields

$$\|v\|_{L^{2^\star}(\Omega)}^{\frac{2}{2^\star}} = \left(\int_\Omega |u|^{\frac{2^\star(\alpha+2)}{2}} \eta^{2^\star} \,\mathrm{d}x \right)^{\frac{2}{2^\star}} \leqslant c \int_\Omega |u|^{\alpha+2} |\nabla \eta|^2 \,\mathrm{d}x ,$$

and the proof is done. □

As a consequence, we have the following corollary:

Corollary 3.21. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of (3.1) where $\mathbb{A}$ is symmetric and fulfils (3.2). Then $u \in L^q_{\mathrm{loc}}(\Omega)$ for every $q \geqslant 1$. Moreover, for every $\alpha \geqslant 0$, every ball $B_r(y) \subset \Omega$ and every $0 < r' < r$, we have the following reverse inequality:*

$$\left(\int_{B_{r'}(y)} |u|^{\frac{2^\star(\alpha+2)}{2}} \,\mathrm{d}x \right)^{\frac{2}{(\alpha+2)2^\star}} \leqslant \frac{c^{\frac{1}{\alpha+2}} (\alpha+2)^{\frac{2}{\alpha+2}}}{(r-r')^{\frac{2}{\alpha+2}}} \left(\int_{B_r(y)} |u|^{\alpha+2} \,\mathrm{d}x \right)^{\frac{1}{\alpha+2}} , \tag{3.9}$$

where $c = c(n, \Lambda) > 0$.

This is a *reverse* inequality in the sense that some L^p-norm is controlled by some L^q-norm, with $p > q$.

Proof. For any K compact subset of Ω, there exists $\eta_K \in C_c^\infty(\Omega)$ with $\eta_K|_K \equiv 1$. Starting with $\alpha = 0$, one can iterate (3.8) with $\eta = \eta_K$ to get $u \in L^{q_k}_{\mathrm{loc}}(\Omega)$ for $q_k := 2^\star k$, $k \in \mathbb{N}$. Since $L^p_{\mathrm{loc}}(\Omega) \subset L^q_{\mathrm{loc}}(\Omega)$, integrability holds for all $q \geqslant 1$.

To derive (3.9), simply take a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $\mathbb{1}_{B_{r'}(y)} \leqslant \eta \leqslant \mathbb{1}_{B_r(y)}$ with $|\nabla \eta| < 2/(r-r')$, and apply Lemma 3.20. □

We are now in a position to prove Lemma 3.18, by iterating Corollary 3.21:

Proof of Lemma 3.18. We fix a ball $B_r(y) \subset \Omega$ and define $(\alpha_i)_{i\in\mathbb{N}}$ as

$$\alpha_i := 2\left(2^\star/2\right)^i - 2 .$$

We also define $(r_i)_{i\in\mathbb{N}}$:

$$r_i = \frac{r}{2} + \frac{r}{2^{i+1}} .$$

Applying Corollary 3.21 with $r = r_i$, $r' = r_{i+1}$ and $\alpha = \alpha_i$ we get

$$M_{i+1} \leqslant c^{1/\beta_i} \beta_i^{2/\beta_i} \left(\frac{r}{2^{i+2}} \right)^{-2/\beta_i} M_i ,$$

where we define $\beta_{i+1} := \alpha_{i+1} + 2 = \frac{2^\star}{2}\beta_i$ and

$$M_i = \left(\int_{B(y,r_i)} |u|^{\beta_i} \,\mathrm{d}x \right)^{1/\beta_i} .$$

By iterating Corollary 3.21, we have

$$M_{i+1} \leqslant c_i M_0 ,$$

where $\lim_{i\to\infty} c_i =: c_\infty < +\infty$. Since

$$\sup_{B_{r/2}(y)} |u| = \lim_{i\to\infty} \left(\int_{B_{r/2}(y)} |u|^i \,\mathrm{d}x \right)^{\frac{1}{i}} \leqslant \lim_{i\to\infty} M_i \leqslant c_\infty M_0 \leqslant c \left(\fint_{B_r(y)} |u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} .$$

□

Lemma 3.18 can be generalized to *weak subsolutions*:

Lemma 3.22. *Let $u \in W^{1,2}(\Omega)$ be a nonnegative weak subsolution of equation (3.1). Then $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, for every ball $B_r(y) \subset \Omega$ and $0 < \sigma < 1$, we have*

$$\sup_{B_{\sigma r}(y)} u \leqslant \frac{c}{(1-\sigma)^{\frac{n}{2}}} \left(\fint_{B_r(y)} u^2 \,\mathrm{d}x \right)^{\frac{1}{2}} ,$$

where $c = c(n, \Lambda) > 0$.

Proof. □

By iterating Lemma 3.22, one can strengthen the result by lowering the exponent on the right-hand side:

Lemma 3.23. *Let $u \in W^{1,2}(\Omega)$ be a nonnegative weak subsolution of equation (3.1). Then $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, for every ball $B_r(y) \subset \Omega$ and $0 < \sigma < 1$, we have*

- $\displaystyle \sup_{B_{\sigma r}(y)} u \leqslant \frac{c}{(1-\sigma)^{\frac{n}{q}}} \left(\fint_{B_r(y)} u^q \,\mathrm{d}x \right)^{\frac{1}{q}}$ *for* $0 < q \leqslant 2$,
- $\displaystyle \sup_{B_{r/2}(y)} u \leqslant c \left(\fint_{B_r(y)} u^q \,\mathrm{d}x \right)^{\frac{1}{q}}$ *for* $q > 2$,

where $c = c(n, \Lambda, q) > 0$.

Proof. We proceed in two steps.

- We first assume $q \leqslant 2$. Take $B_r(y) \subset \Omega$ and some $0 < \sigma < 1$. Define $(\sigma_i)_{i \in \mathbb{N}}$ as

$$\sigma_i := 1 - \frac{1-\sigma}{2^i} ,$$

so that σ_i varies monotonically from σ to 1 as i varies from 0 to ∞. We can now use Lemma 3.22 with $r = \sigma_{i+1} r$ and $\sigma = \sigma_i / \sigma_{i+1}$ and get

$$\begin{aligned} M_i := \sup_{B_{\sigma_i r}(y)} u &\leqslant \frac{c}{\left(1 - \frac{\sigma_i}{\sigma_{i+1}}\right)^{\frac{n}{2}}} \left(\fint_{B_{\sigma_{i+1} r}(y)} u^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leqslant \frac{c}{\left(1 - \frac{\sigma_i}{\sigma_{i+1}}\right)^{\frac{n}{2}}} \left(\fint_{B_{\sigma_{i+1} r}(y)} u^q \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\sup_{B_{\sigma_{i+1} r}(y)} u \right)^{\frac{2-q}{2}} , \\ &\leqslant \frac{c}{\left(1 - \frac{\sigma_i}{\sigma_{i+1}}\right)^{\frac{n}{2}}} \left(\fint_{B_{\sigma_{i+1} r}(y)} u^q \,\mathrm{d}x \right)^{\frac{1}{2}} M_{i+1}^{\frac{2-q}{2}} . \end{aligned}$$

Iterating this inequality, we get the result.

- Now assume $q > 2$. We start from the above result for $q = 2$:

$$\begin{aligned} \sup_{B_{\sigma r}(y)} u &\leqslant c \left(\fint_{B_r(y)} u^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &= c \left(\fint_{B_r(y)} u^2 \,\mathrm{d}x \right)^{\frac{1}{q}\frac{q}{2}} , \end{aligned}$$

where the second factor is bounded by 1 and $x \mapsto x^{\frac{q}{2}}$ is convex, so, by JENSEN's inequality [7, Theorem 2.2], we have:

$$\leqslant c \left(\fint_{B_r(y)} u^q \, \mathrm{d}x \right)^{\frac{1}{q}} .$$

This completes the proof.

□

Remark 3.24. *On can derive a similar result in the case $n = 2$.*

HARNACK's inequality: inf Here, we prove the following:

Lemma 3.25. *Let $u \in W^{1,2}(\Omega)$ be a nonnegative weak solution of (3.1), where $\mathbb{A}$ fulfils (3.2). Then there are $q = q(n, \Lambda) > 0$ and $c = c(n, \Lambda) > 0$ such that, for every ball $B_{2r}(y) \subset \Omega$, we have*

$$\inf_{B_{r/2}(y)} u \geqslant c \left(\fint_{B_r(y)} u^q \, \mathrm{d}x \right)^{\frac{1}{q}} . \tag{3.10}$$

Remark 3.26. *Given $\varepsilon > 0$, we can assume $u \geqslant \varepsilon$ in Ω by replacing u by $u + \varepsilon$.*

The key point in the proof is the fact that $\log u$ is a function of *bounded mean oscillation* (BMO).

Lemma 3.27. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of (3.1), where A fulfils (3.2). Suppose that $u \geqslant \varepsilon$ in Ω for some $\varepsilon > 0$. Then, for any $q > 0$, there is a constant $c = c(n, \Lambda, q) > 0$ such that the following holds:*

$$\inf_{B_{r/2}(y)} u \geqslant c \left(\fint_{B_r(y)} u^{-q} \, \mathrm{d}x \right)^{-\frac{1}{q}} .$$

Proof. We claim that u^{-1} is a subsolution of (3.1). Indeed, first it is easy to show that $u^{-1} \in H^1_{\mathrm{loc}}(\Omega)$. Second, for any nonnegative $\eta \in C_c^\infty(\Omega)$, define $\phi := \eta u^{-2}$. We test (3.1) with ϕ to get

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla\phi, \mathbb{A}(x)\nabla u \rangle \, \mathrm{d}x \\ &= \int_\Omega u^{-2} \langle \nabla\eta, \mathbb{A}(x)\nabla u \rangle \, \mathrm{d}x - 2 \int_\Omega \eta u^{-3} \langle \nabla u, \mathbb{A}\nabla u \rangle \, \mathrm{d}x \end{aligned}$$

The last term is nonpositive, so we have

$$\int_\Omega \langle \nabla\eta, \mathbb{A}(x)\nabla v \rangle = - \int_\Omega u^{-2} \langle \nabla\eta, \mathbb{A}(x)\nabla u \rangle \, \mathrm{d}x \leqslant 0 ,$$

and u^{-1} is a subsolution. We can then apply Lemma 3.23 with $\sigma = ½$ and get

$$\inf_{B_{r/2}(y)} u \geqslant c \left(\fint_{B_r(y)} u^{-q} \, \mathrm{d}x \right)^{-\frac{1}{q}} .$$

□

Next, we show that $\log u$ is BMO:

Lemma 3.28. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of (3.1), where $\mathbb{A}$ fulfils (3.2). Suppose that $u \geqslant \varepsilon$ in Ω for some $\varepsilon > 0$. Then, for every ball $B_{2r}(y) \subset \Omega$, we have*

$$\int_{B_r(y)} |\nabla v|^2 \, \mathrm{d}x \leqslant c r^{n-2} ,$$

where $v = \log u$ and $c = c(n, \Lambda) > 0$.

Proof. Fix $\eta \in C_c(\Omega)$ and let $\phi = \eta^2 u^{-1}$. We again test (3.1) against ϕ and get that

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla\phi, \mathbb{A}(x)\nabla u\rangle \,\mathrm{d}x \\ &= -\int_\Omega \eta^2 u^{-2}\langle \nabla u, \mathbb{A}\nabla u\rangle \,\mathrm{d}x + 2\int_\Omega \eta u^{-1}\langle \nabla\eta, \mathbb{A}(x)\nabla u\rangle \,\mathrm{d}x \end{aligned}$$

which, after using the ellipticity and continuity of $\mathbb{A}$ as well as the Cauchy-Schwarz inequality like before, gives

$$\int_\Omega |\nabla v|^2\eta^2 \,\mathrm{d}x \leqslant 4\Lambda^2 \int_\Omega |\nabla\eta|^2 \,\mathrm{d}x\,.$$

The result follows by choosing $\eta \in C_c^\infty(\Omega)$ such that $\mathbb{1}_{B_{2r}(y)} \leqslant \eta \leqslant \mathbb{1}_{B_r(y)}$ and $|\nabla\eta| \leqslant 2/r$. □

Proof of Lemma 3.25. On any ball $B_{2r}(y) \subset \Omega$, we can use the Poincaré-Wirtinger inequality (Theorem A.3) and Lemma 3.28 to get

$$\int_{B_r(y)} |v - v_{B_r(y)}|^2 \,\mathrm{d}x \leqslant c(n)\, r^2 \int_{B_r(y)} |\nabla v|^2 \leqslant c(n,\Lambda)\, r^2 r^{n-2}\,,$$

which yields

$$\fint_{B_r(y)} |v - v_{B_r(y)}|^2 \,\mathrm{d}x \leqslant c(n,\Lambda)\,.$$

Thus, $v = \log u \in \mathcal{L}^{2,n}(\Omega) \cong \mathrm{BMO}(\Omega)$. We can then use the John-Nirenberg inequality Lemma 2.23, which yields

$$\fint_{B_r(y)} \exp\left(c_1|v - v_{B_r(y)}|\right) \,\mathrm{d}x \leqslant c_2$$

for $c_1 = c_1(n,\Lambda) > 0$ and $c_2 = c_2(n,\Lambda) > 0$. Then we have

$$\begin{aligned} \fint_{B_r(y)} u^{c_1} \,\mathrm{d}x \fint_{B_r(y)} u^{-c_1} \,\mathrm{d}x &= \fint_{B_r(y)} \exp\left(c_1(v - v_{b_r(y)})\right) \,\mathrm{d}x \fint_{B_r(y)} \exp\left(c_1(v_{b_r(y)} - v)\right) \,\mathrm{d}x \\ &\leqslant \left(\fint_{B_r(y)} \exp\left(c_1|v - v_{B_r(y)}|\right) \,\mathrm{d}x\right)^2 \\ &\leqslant (c_2)^2\,. \end{aligned}$$

This, together with Lemma 3.27, proves (3.10) with $q = c_1$. This finishes the proof. □

Proof of the Harnack inequality Theorem 3.15. Simply combine Lemmata 3.23 and 3.25. □

We are now able to conclude with the Hölder continuity of solutions:

Proof of Theorem 3.17. Thanks to Lemma 3.18, we know that $\sup u$ and $\inf u$ are locally bounded, so we only need an estimate for the Hölder semi-norm. We pick $B_R(x_0) \Subset \Omega$ and define $m(x_0,R) := \inf_{B_R(x_0)} u$ and $M(x_0,R) := \sup_{B_R(x_0)} u$. Next, we apply Theorem 3.15 to the (nonnegative) functions

$$M(x_0,R) - u \quad \text{and} \quad u - m(x_0,R)\,,$$

and get

$$\begin{aligned} M(x_0,R) - m(x_0,R/2) &\leqslant c(n,\Lambda)\left(M(x_0,R) - M(x_0,R/2)\right)\,, \\ M(x_0,R/2) - m(x_0,R) &\leqslant c(n,\Lambda)\left(m(x_0,R/2) - m(x_0,R)\right)\,, \end{aligned}$$

where the constants are the same on both lines. Summing this up we obtain

$$\operatorname{osc}_{B_R(x_0)} u + \operatorname{osc}_{B_{R/2}(x_0)} u \leqslant c(n,\Lambda)\left(\operatorname{osc}_{B_R(x_0)} u - \operatorname{osc}_{B_{R/2}(x_0)} u\right)\,.$$

Hence, we have

$$\operatorname{osc}_{B_{R/2}(x_0)} u \leqslant 2^{-\alpha} \operatorname{osc}_{B_R(x_0)} u \,,$$

for some $\alpha \in (0, 1]$ satisfying

$$2^{-\alpha} \geqslant \frac{c(n, \Lambda) - 1}{c(n, \Lambda) + 1} \,.$$

Note that α does not depend on x_0. We can iterate this estimate and find

$$\operatorname{osc}_{B_{2^{-j}R}(x_0)} u \leqslant 2^{-j\alpha} \operatorname{osc}_{B_R(x_0)} u \qquad \text{for all } j \in \mathbb{N} \,.$$

For $r \in (0, R]$, there is a unique $j_0 \in \mathbb{N}$ such that

$$2^{-j_0-1} R < r \leqslant 2^{-j_0} R \,,$$

from which we get

$$\operatorname{osc}_{B_r(x_0)} u \leqslant \operatorname{osc}_{B_{2^{-j_0}R}(x_0)} u \leqslant 2^{-j_0\alpha} \operatorname{osc}_{B_R(x_0)} u \leqslant 2^{\alpha} \left(\frac{r}{R}\right)^{\alpha} \operatorname{osc}_{B_R(x_0)} u \,.$$

This gives the result, with a similar argument as in the proof of Theorem 3.2, p.18. □

The theorem of De Giorgi-Nash-Moser can be generalized as follows:

Theorem 3.29. *Let $1 < p < \infty$ and let $u \in W^{1,p}(\Omega)$ be a weak solution of*

$$-\operatorname{div} a(Du, u, x) = a_0(Du, u, x) \qquad \text{in } \Omega \,,$$

where $a : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $a_0 : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ are such that:

- *a and a_0 are Carathéodory functions,* i.e. *$a : (z, u, x) \mapsto a(z, u, x)$ and $a_0 : (z, u, x) \mapsto a_0(z, u, x)$ are measurable w.r.t. x for all $(z, u) \in \mathbb{R}^n \times \mathbb{R}$ and continuous w.r.t. (z, u) for almost all $x \in \Omega$.*
- *There exists $L \geqslant 1$ such that*

$$\begin{aligned} |a(z, u, x)| &\leqslant \Lambda\,(1 + |z|)^{p-1} \\ |a_0(z, u, x)| &\leqslant \Lambda\,(1 + |z|)^{p-1} \\ \langle a(z, u, x), z \rangle &\geqslant |z|^p \end{aligned}$$

  hold for all $(z, u) \in \mathbb{R}^n \times \mathbb{R}$ and almost every $x \in \Omega$.

Then, there exists $0 < \alpha = \alpha(n, p, \Lambda)$ such that $u \in C^{0,\alpha}(\Omega)$.

With the following counter-example, we will see that the Hölder continuity does not hold for every exponent $\alpha > 0$.

Example 3.30. *We consider a ball $B_1 \subset \mathbb{R}^n$, where $n \geqslant 2$. Let $u : B_1 \to \mathbb{R}$ defined by $u(x) = x^1 |x|^{\alpha-1}$, for some $\alpha \in (0, 1)$. Then we have the following:*

- *$u \in W^{1,2}(B_1) \cap C^{0,\alpha}(\bar{B}_1)$,*
- *$u \notin C^{0,\beta}(B_1)$ for $\beta > \alpha$,*
- *u is a weak solution to the equation $\operatorname{div}(\mathbb{A}\nabla u) = 0$ in B_1, where the matrix $\mathbb{A}$ has measurable, bounded, elliptic coefficients defined by*

$$a^{ij}(x) := \delta^{ij} + \frac{(1-\alpha)(n-1+\alpha)}{\alpha\,(n-2+\alpha)} \frac{x^i x^j}{|x|^2} \qquad \text{for } 1 \leqslant i, j \leqslant n \,.$$

Proof. The optimal Hölder continuity of u with exponent α is clear. We have that $|u| \leqslant |x|^\alpha \in L^q(B_1)$ for $q = \infty$. u also has a classical derivative outside of the origin, which is given by

$$D_j u(x) = \delta^{1j}|x|^{\alpha-1} + (\alpha - 1)\, x^1 x^j |x|^{\alpha-3}\,.$$

For $1 \leqslant p < n/(1-\alpha) < q$, we have that $|D_j u(x)|^p \leqslant C|x|^{(\alpha-1)p} \in L^1(B_1 \setminus \{0\})$, so that $Du \in L^p(B_1 \setminus \{0\})$.

Let us now prove that $\{0\}$ satisfies (2.1). Define the sequence $(\tilde{\psi}_j)_{j\in\mathbb{N})} \subset W^{1,n}(\mathbb{R}^n, [0,1])$ as follows:

$$\tilde{\psi}_j(x) := \begin{cases} 1 & \text{if } |x| \leqslant \exp(-\exp(j+1))\,, \\ \log(-\log|x|) - j & \text{if } \exp(-\exp(j+1)) < |x| < \exp(-\exp(j))\,, \\ 0 & \text{if } |x| \geqslant \exp(-\exp(j))\,. \end{cases}$$

These functions are rotationally symmetric with compact support in a ball whose radius vanishes as $j \to \infty$. They are also equal to one close to the origin. We can compute

$$\begin{aligned} \int_{\mathbb{R}^n} |D\tilde{\psi}_j|^n \,\mathrm{d}x &= c(n) \int_{\exp(-\exp(j+1))}^{\exp(-\exp(j))} |\log r|^n r^{-1} \,\mathrm{d}r \\ &= c(n)(\exp(j)^{1-n} - \exp(j+1)^{1-n}) \to 0 \text{ as } j \to \infty\,. \end{aligned}$$

This shows that $\|\tilde{\psi}_j\|_{W^{1,n}(\mathbb{R}^n)} \to 0$. By regularization by suitable mollifying kernels, we can obtain a sequence of functions $(\psi_j)_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, [0,1])$ with the same properties. By Hölder's inequality, (2.1) holds for all $q' \in [1, n]$.

We can thus apply Lemma 2.9, from which we get that $u \in W^{1,p}(B_1)$ for all $p \in [1, n/(1-\alpha))$, and its weak derivative is given by the expression above.

It remains to check that u is a weak solution of the given equation. For $x \neq 0$, we have

$$\sum_{1\leqslant j\leqslant n} a^{ij}(x) D_j u(x) = \delta^{1i}|x|^{\alpha-1} + \frac{1-\alpha}{n-2+\alpha} x^1 x^i |x|^{\alpha-3}\,,$$

from which we get

$$\begin{aligned} \sum_{1\leqslant i,j\leqslant n} D_i\left(a^{ij}(x) D_j u\right) &= D_1|x|^{\alpha-1} + \frac{1-\alpha}{n-2+\alpha} \sum_{1\leqslant i\leqslant n} D_i(x^1 x^i |x|^{\alpha-3}) \\ &= x^1|x|^{\alpha-3}\left((\alpha-1)\,\frac{1-\alpha}{n-2+\alpha}\,(1+n+\alpha-3)\right) \\ &= 0\,. \end{aligned}$$

Again using Lemma 2.9, we get that u is a weak solution in the whole of B_1. $\square$

4 Elliptic systems

We follow Beck [1, Section 4.1].

In this section, we consider the vectorial case, that is when u is vector-valued: $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ for $m > 1$. Instead of equation (3.1), we have a *system* of equations:

$$\operatorname{div}(\mathbb{A}(Du(x), u(x), x)Du(x)) = D_i\left(A_{st}^{ij}(Du(x), u(x), x)D_j u^t\right) = \sum_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant t\leqslant m}} \frac{\partial}{\partial x^i}\left(A_{st}^{ij}\frac{\partial}{\partial x^j}u^t\right) = 0 \tag{4.1}$$

where

$$\mathbb{A} := \left(A_{st}^{ij}\right)_{1\leqslant s,t\leqslant m}^{1\leqslant i,j\leqslant n} : \mathbb{R}^{mn}\times\mathbb{R}^m\times\Omega \to \mathbb{R}^{mn\times mn}.$$

As before, we assume $\mathbb{A}$ to meet the Carathéodory conditions, *i.e.* we assume $\mathbb{A}(\cdot,\cdot,x)$ to be continuous for almost all x and $\mathbb{A}(z,u,\cdot)$ to be measurable for all z, u.

Similarly to the scalar case, we have the notion of ellipticity for $\mathbb{A}$:

Definition 4.1 (Ellipticity)**.** *We say that $\mathbb{A}$ is* elliptic *if there exists $\lambda > 0$ such that the inequality*

$$\lambda|\xi|^2 \leqslant \sum_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant s,t\leqslant m}} A_{st}^{ij}(z,u,x)\,\xi_i^s\xi_j^t \tag{4.2}$$

holds for all $\xi \in \mathbb{R}^{mn}$, all z, u and almost all x.

The condition (4.2) is also known as the Legendre ellipticity condition, or *very strong* ellipticity condition.

Definition 4.2 (Weak solution)**.** *We say that $u \in W^{1,2}(\Omega,\mathbb{R}^m)$ is a* weak solution *of (4.1) if the following holds for all $\phi \in W_0^{1,2}(\Omega,\mathbb{R}^m)$:*

$$\int_\Omega A_{st}^{ij}(Du(x), u(x), x)\,D_j u^t D_i\phi^s\,\mathrm{d}x = 0\,.$$

4.1 Counterexamples to regularity

In this section, we will need Lemma 2.9.

The goal is to see that there exists discontinuous weak solutions and that this occurs also for relatively simple linear systems. Let us start with the case $m = n$, $\Omega = B_1$ by considering the simple function

$$u(\alpha, x) := |x|^{-\alpha}x\,,$$

where $\alpha \in [1, n/2)$. It is only discontinuous at the origin, and actually belongs to the Sobolev space $W^{1,2}(B_1,\Omega)$. Away from the origin, its weak derivative is given by

$$D_i u^s(\alpha, x) = |x|^{-\alpha}\delta_i^s - \alpha|x|^{-\alpha-2}x^i x^s\,,$$

for $1 \leqslant i, s \leqslant n$. We also have the following identities:

$$\sum_{i=1}^n D_i u^i(\alpha,x) = \operatorname{tr}(Du(\alpha,x)) = (n-\alpha)|x|^{-\alpha},$$

$$\sum_{i,s=1}^n x^i x^s D_i u^s(\alpha,x) = (1-\alpha)|x|^{2-\alpha}\,,$$

$$\sum_{i,s=1}^n (D_i u^s(\alpha,x))^2 = |Du(\alpha,x)|^2 = (n - 2\alpha + \alpha^2)|x|^{-2\alpha}\,.$$

4.1.1 The counterexample of De Giorgi

We start by introducing a family of bilinear forms $\mathbb{B}(b_1, b_2)$ on $\mathbb{R}^{n\times n}$, which are thus defined:

$$B^{ij}_{st}(b_1, b_2, x) := \delta_{st}\delta^{ij} + \left(b_1\delta^s_i + b_2\frac{x^i x^s}{|x|^2}\right)\left(b_1\delta^t_j + b_2\frac{x^j x^t}{|x|^2}\right),$$

for $x \neq 0$, $1 \leqslant i, j, s, t \leqslant n$, where b_1, b_2 are real parameters. Note how the factors δ^s_i and $x^i x^s$ appear both in the definition of $\mathbb{B}$ and in the expression for $D_i u^s$. In the following, we write

$$\left(\mathbb{B}(b_1, b_2, x)\,\xi, \bar{\xi}\right) = B^{ij}_{st}(b_1, b_2, x)\,\xi^s_i\bar{\xi}^t_j = \sum_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant s,t\leqslant n}} B^{ij}_{st}(b_1, b_2, x)\,\xi^s_i\bar{\xi}^t_j,$$

for all $\xi, \bar{\xi} \in \mathbb{R}^{n\times n}$. From its definition, we see that $\mathbb{B}(b_1, b_2, x)$ is elliptic and bounded, *i.e.* for every $b_1, b_2 \in \mathbb{R}\times\mathbb{R}$, there exists $c(b_1, b_2) > 0$ such that the following inequality holds:

$$|\xi|^2 \leqslant \left(\mathbb{B}(b_1, b_2, x)\,\xi, \xi\right) \leqslant c(b_1, b_2)|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^{n\times n},\ x \neq 0. \tag{4.3}$$

We can choose the parameters b_1, b_2 in such a way that, for each $1 \leqslant \alpha < n/2$, $u(\alpha, \cdot)$ as defined above is a weak solution of (4.1). Let us compute

$$\begin{aligned}\left(\mathbb{B}(b_1, b_2, x)Du(\alpha, x)\right)^j_t &= D_j u^t(\alpha, x) + \left(b_1\sum_{i=1}^n D_i u^i(\alpha, x) + b_2\sum_{i,s=1}^n \frac{x^i x^s}{|x|^2}D_i u^s(\alpha, x)\right)\left(b_1\delta^j_t + b_2\frac{x^j x^t}{|x|^2}\right)\\ &= \quad [b_1(b_1(n-\alpha) + b_2(1-\alpha)) + 1]|x|^{-\alpha}\delta^j_t\\ &\quad + [b_2(b_1(n-\alpha) + b_2(1-\alpha)) - \alpha]|x|^{-\alpha-2}x^j x^t.\end{aligned}$$

Observe that

$$\sum_{j=1}^n D_j(|x|^\alpha\delta^j_t) = -\alpha|x|^{-\alpha-2}x^t$$

and

$$\sum_{j=1}^n D_j\left(|x|^{-\alpha-2}x^j x^t\right) = (n-1-\alpha)|x|^{-\alpha-2}x^t,$$

so that

$$\sum_{j=1}^n D_j(\mathbb{B}(b_1, b_2, x)Du(\alpha, x))^j_t = 0$$

holds all $x \neq 0$ if α, b_1 and b_2 fulfil the equation

$$\alpha[b_1(b_1(n-\alpha) + b_2(1-\alpha)) + 1] = (n-1-\alpha)[b_2(b_1(n-\alpha) + b_1(1-\alpha)) - \alpha] \tag{4.4}$$

$$\Leftrightarrow \alpha^2[(b_1+b_2)^2 + 1] - \alpha n[(b_1+b_2)^2 + 1] + (n-1)b_2(b_2 + b_1 n) = 0. \tag{4.5}$$

In this case, the function $u(\alpha, \cdot)$ is a weak solution of (4.1) in B_1, by Lemma 2.9. For $\alpha = 1$, this yields a bounded, discontinuous weak solution, and for $1 < \alpha < n/2$, we even have unbounded, discontinuous weak solutions. The choice $b_1 = n-2$, $b_2 = n$, with $1 < \alpha < n/2$ chosen according to (4.5) was proposed by De Giorgi:

Example 4.3 (De Giorgi)**.** *Let $n \geqslant 3$ and $u : \mathbb{R}^n \supset B_1 \longrightarrow \mathbb{R}^n$ be given by*

$$u(\alpha, x) = |x|^{-\alpha}x \quad \textit{for } \alpha := \frac{n}{2}\left(1 - ((2n-2)^2 + 1)^{-\frac{1}{2}}\right).$$

Then $u \in W^{1,2}(B_1, \mathbb{R}^n)$ is an unbounded weak solution of the elliptic system

$$\operatorname{div}(\mathbb{B}(n-2, n, x)Du(\alpha)) = 0 \quad \textit{in } B_1.$$

Since $\mathbb{B}$ is bounded and elliptic with measurable entries, it satisfies all the conditions of Theorem 3.2. The counterexample of De Giorgi then highlights the fact that, in the vectorial case, one cannot expect Hölder regularity of all (even bounded) weak solutions. One cannot even expect local boundedness, so that Theorem 3.2 cannot be extended to functions with values in $\mathbb{R}^{m=n}$ in the case $n \geqslant 3$.

4.1.2 The counterexample of Giusti and Miranda

The entries of $\mathbb{B}$, as discussed in De Giorgi's counterexample, are discontinuous at the origin. It is natural to wonder what happens in the case where there coefficients $a(u,x)$ are regular enough. One needs to distinguish between two cases:

- the linear case, *i.e.* $\mathbb{A}(u,x) = \mathbb{A}(x)$, in which case the continuity or smoothness of the coefficients actually implies the continuity or smoothness of weak solutions. In other words, a weak solution can only be discontinuous if the a is.
- the quasilinear case, where $\mathbb{A}$ is allowed to depend on u. Here, for a system with smooth coefficients, Giusti and Miranda constructed an irregular weak solution. Their counterexample, built starting from Example 4.3, is an elliptic system whose coefficients depend smoothly on the solution and which admits a (bounded) discontinuous weak solution.

In the following, we consider the function $u(1,x) = x/|x|$, which is a weak solution of the system

$$\operatorname{div}\big(\mathbb{B}(1, 2/(n-2), x)\, Du\big) = 0 \quad \text{in } B_1,$$

where $\mathbb{B}$ is defined in Section 4.1.1, with $b_1 = 1, b_2 = 2/(n-2)$ and $\alpha = 1$. Then, we can replace all occurrences of $x^i/|x|$ in the expression for $\mathbb{B}$ by u^i. Noting that $|u(x)| = 1$ for $x \neq 0$, we obtain

$$\tilde{B}^{ij}_{st}(u) = \delta_{st}\delta^{ij} + \left(\delta^i_s + \frac{4}{n-2}\frac{u^i u^s}{1+|u|^2}\right)\left(\delta^t_j + \frac{4}{n-2}\frac{u^j u^t}{1+|u|^2}\right),$$

for all $1 \leqslant i,j,s,t \leqslant n$ and all $u \in \mathbb{R}^n$. These coefficients are smooth in u, elliptic and bounded, and we end up with the counterexample of Giusti and Miranda:

Example 4.4. *Let $n \geqslant 3$ and $u : \mathbb{R}^n \supset B_1 \to \mathbb{R}^n$ be given by $u(x) = x/|x|$. Then, $u \in W^{1,2}(B_1, \mathbb{R}^n) \cap L^\infty(B_1, \mathbb{R}^n)$ and u is a discontinuous weak solution of the elliptic system*

$$\operatorname{div}(\tilde{\mathbb{B}}(u)Du) = 0 \quad \text{in } B_1\,.$$

Remark 4.5. *In the case $n = 2$, all weak solutions are continuous, and their gradient has the same regularity of the coefficients, see* e.g. *[1] and the next section.*

4.2 The hole-filling technique

Caccioppoli's inequality may be used to obtain a decay estimate for the Dirichlet integral of weak solutions of linear elliptic systems. Here we show how to do this by the *hole-filling* technique of Widman (see [15]). As a consequence we obtain Hölder continuity for the solutions of elliptic systems with bounded coefficients in dimension 2.

Let $\Omega \subset \mathbb{R}^n$ with smooth boundary and $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a weak solution to the following elliptic linear elliptic system:

$$-D_\alpha\left(A^{\alpha\beta}_{ij} D_\beta u^j\right) = 0 \quad \text{in } \Omega\,, \tag{4.6}$$

where the matrix $\mathbb{A} = (A^{\alpha\beta}_{ij})^{1\leqslant\alpha,\beta\leqslant n}_{1\leqslant i,j\leqslant m} \in L^\infty(\Omega, \mathbb{R}^{mn\times mn})$ satisfies the condition (4.2). Take $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$. Test equation (4.6) against the function $(u-\xi)\eta^2$, where $\xi \in \mathbb{R}^m$ and η is a nonnegative cut-off function with $\mathbb{1}_{B_{R/2}(x_0)} \leqslant \eta \leqslant \mathbb{1}_{B_R(x_0)}$ and $\|D\eta\|_{L^\infty} \leqslant 4/R$. We get:

$$\begin{aligned} 0 = \int_\Omega A^{\alpha\beta}_{ij} D_\beta u^j D_\alpha((u^i - \xi^i)\eta^2)\,\mathrm{d}x &= \int_\Omega A^{\alpha\beta}_{ij} D_\beta u^j [\eta^2 D_\alpha u^i + 2\eta(u^i-\xi^i)D_\alpha\eta]\,\mathrm{d}x\,,\\ &\geqslant \lambda \int_\Omega \eta^2 |Du|^2\,\mathrm{d}x + 2\int_\Omega A^{\alpha\beta}_{ij}\eta D_\beta u^j (u^i-\xi^i) D_\alpha\eta\,\mathrm{d}x\,, \end{aligned}$$

now, use the boundedness of $\mathbb{A}$ and get:

$$\int_\Omega \eta^2 |Du|^2 \,\mathrm{d}x \leqslant c \int_\Omega \eta |Du||D\eta||u-\xi| \,\mathrm{d}x \,,$$

where $c > 0$ is independent of u and R. At this point, we use the properties of η, the POINCARÉ inequality and YOUNG's inequality with ε to bound the right-hand side from above by

$$\frac{1}{2}\int_\Omega \eta^2 |Du|^2 \,\mathrm{d}x + \frac{c_1}{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} |u-\xi|^2 \,\mathrm{d}x \,. \tag{4.7}$$

By choosing

$$\xi = \fint_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} u \,\mathrm{d}x \,,$$

we can use the following POINCARÉ-type inequality:

$$\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} |u-\xi|^2 \,\mathrm{d}x \leqslant c_2 R^2 \int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} |Du|^2 \,\mathrm{d}x \tag{4.8}$$

to find

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \,\mathrm{d}x \leqslant c \int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} |Du|^2 \,\mathrm{d}x \,,$$

where $c > 0$, importantly, does not depend on R (nor u). We now *fill the hole* on the right-hand side by adding c times the left-hand side to both sides, and get:

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \,\mathrm{d}x \leqslant \frac{c}{c+1} \int_{B_R(x_0)} |Du|^2 \,\mathrm{d}x \,, \tag{4.9}$$

and then

$$\int_{B_{2^{-k}R}(x_0)} |Du|^2 \,\mathrm{d}x \leqslant \left(\frac{c}{c+1}\right)^k \int_{B_R(x_0)} |Du|^2 \,\mathrm{d}x$$

for all $k \geqslant 1$. This yields the existence of some $\alpha = \alpha(\lambda, \Lambda) > 0$ such that

$$\int_{B_\rho(x_0)} |Du|^2 \,\mathrm{d}x \leqslant c_1 \rho^{2\alpha} \,.$$

When $n = 2$, we can then use MORREY's Theorem 2.20 to get $u \in C^{0,\alpha}(\Omega, \mathbb{R}^m)$.

Remark 4.6. (4.8) *can be proven by choosing $R = 1$ and making use of the* POINCARÉ-WIRTINGER *inequality. The general result is obtained by rescaling.*

Another consequence of (4.9) is that *entire solutions* (4.6), *i.e.* solutions of (4.6) in all of $\mathbb{R}^n$, with finite energy,

$$\|Du\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |Du|^2 \,\mathrm{d}x < \infty \,,$$

are constant: for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\|Du\|^2_{L^2(\mathbb{R}^n)} - \varepsilon < \int_{B_{r_\varepsilon/2}(0)} |Du|^2 \;\mathrm{d}x \leqslant \frac{c}{c+1} \int_{B_{r_\varepsilon}(0)} |Du|^2 \,\mathrm{d}x \leqslant \frac{c}{c+1} \|Du\|^2_{L^2(\mathbb{R}^n)} \,,$$

which can only hold (as $\varepsilon \to 0$) if $Du = 0$ almost everywhere. Consider now an entire solution u of (4.6) is dimension $n = 2$. Suppose it is globally bounded; then from (4.7) with $\xi = 0$ we get

$$\int_{B_R(0)} |Du|^2 \,\mathrm{d}x \leqslant \frac{c}{R^2} \int_{B_{2R}(0)} |u|^2 \,\mathrm{d}x \leqslant c_1 \sup_{\mathbb{R}^2} |u|^2 \,.$$

Hence, u has finite energy. Therefore we have the following

Theorem 4.7. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ be a bounded solution of the elliptic system (4.6) with $\Omega = \mathbb{R}^2$. Then u is constant.*

Remark 4.8. *If $\mathbb{A}$ is continuous (or constant), this result also holds if one weakens the ellipticity condition (4.2) a little. See for example [4, Section 4.4].*

Notation

n	the space dimension
B_r	the open ball centred at 0 and of radius r
u_-, u_+	$\min(0,u)$, $\max(0,u)$
$C_c^\infty(\Omega)$	the space of smooth functions with compact support in Ω
$\langle \cdot, \cdot \rangle$	the canonical scalar product in $\mathbb{R}^n$
a.e.	almost everywhere, almost every
$\mathbb{1}_A$	the indicator function of the set A
f_A	the average of the function f over the set A, *i.e.* $\lvert A\rvert^{-1} \int_A f \, \mathrm{d}x$.

A Appendix

Theorem A.1 (LEBESGUE differentiation theorem). *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then, for almost every $x_0 \in \mathbb{R}^n$, we have*

$$\lim_{r \searrow 0} \fint_{B_r(x_0)} |f(x) - f(x_0)| \, \mathrm{d}x = 0 .$$

Such a point x_0 is called a LEBESGUE point *of f. In particular, it holds*

$$\lim_{r \searrow 0} \fint_{B_r(x_0)} f(x) \, \mathrm{d}x = f(x_0) .$$

As a consequence, we have a version for L^p spaces:

Theorem A.2 (LEBESGUE differentiation theorem for L^p spaces). *Let $1 \leqslant p < \infty$ and $f \in L^p_{\text{loc}}(\mathbb{R}^n)$. Then, for almost every $x_0 \in \mathbb{R}^n$, we have*

$$\lim_{r \searrow 0} \fint_{B_r(x_0)} |f(x) - f(x_0)|^p \, \mathrm{d}x = 0 .$$

*Such a point x_0 is called a p-*LEBESGUE point *of f. In particular, it holds*

Theorem A.3 (POINCARÉ-WIRTINGER). *Let $1 \leqslant p < \infty$. For every bounded and connected domain Ω with the extension property* (e.g. *with* LIPSCHITZ *boundary) there exists $c = c(n, p, \Omega)$ such that for each $u \in W^{1,p}(\Omega)$ we have*

$$\int_\Omega |u - u_\Omega|^p \, \mathrm{d}x \leqslant c \int_\Omega |Du|^p \, \mathrm{d}x .$$

When Ω is a ball of radius r or a cube of side length r, we can take $c(n, p, \Omega) = c(n, p) r^p$.

Proof. Assume that the assertion does not hold. We can then find a sequence $(u_j)_{j \in \mathbb{N}}$ with

$$\int_\Omega |Du_j|^p \, \mathrm{d}x \to 0 , \quad (u_j)_\Omega = 0 , \quad \int_\Omega |u|^p \, \mathrm{d}x = 1 .$$

By RELLICH's and BANACH-ALAOGLU's theorems, we can extract a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such that

$$u_{n_k} \xrightarrow{L^p} u , \quad u_{n_k} \overset{W^{1,p}}{\rightharpoonup} u .$$

In particular, $Du = 0$, *i.e.* u is constant, $\|u\|_{L^p(\Omega)} = 1$ and $u_\Omega = 0$, which is a contradiction. The claim on c follows by scaling: consider w.l.o.g. $u \in W^{1,p}(B_r(0))$, so that $\tilde{u} : x \mapsto u(rx) \in W^{1,p}(B_1(0))$. The result follows by writing the POINCARÉ-WIRTINGER inequality for $\tilde{u}$ and using the identities

$$\fint_{B_r(0)} u(x) \, \mathrm{d}x = \fint_{B_1(0)} \tilde{u}(x) \, \mathrm{d}x ,$$

$$\int_{B_r(0)} |u(x) - u_{B_r(0)}|^p \, \mathrm{d}x = r^{-d} \int_{B_1(0)} |\tilde{u}(x) - \tilde{u}_{B_1(0)}|^p \, \mathrm{d}x ,$$

$$\int_{B_r(0)} |Du(x)|^p \, \mathrm{d}x = r^{-d} r^p \int_{B_1(0)} |D\tilde{u}(x)|^p \, \mathrm{d}x .$$

□

Theorem A.4 (MARKOV's inequality). *Let (X, Σ, μ) be a measure space and $a > 0$. If f a measurable function with values in $\mathbb{R} \cup \{-\infty, +\infty\}$, then the following inequality holds:*

$$\mu\big(\{x \in X : |f(x)| \geqslant a\}\big) \leqslant \frac{1}{a} \int_X |f| \, \mathrm{d}\mu .$$

Proof. We have that

$$a \cdot \mathbb{1}_{\{x \in X \,:\, |f(x)| \geqslant a\}} \leqslant |f| \cdot \mathbb{1}_{\{x \in X \,:\, |f(x)| \geqslant a\}} ,$$

which, after integration over X, yields

$$a \cdot \mu\left(\{x \in X \;:\; |f(x)| \geqslant a\}\right) \leqslant \int_X |f| \cdot \mathbb{1}_{\{x \in X \,:\, |f(x)| \geqslant a\}} \,\mathrm{d}\mu \leqslant \int_X |f| \,\mathrm{d}\mu . \qquad \square$$

By the same token, we also have a corresponding result for f^p, which is also known as the MARKOV-CHEBYSHEV inequality:

Theorem A.5 (CHEBYSHEV's inequality)**.** *If f a measurable function with values in $\mathbb{R} \cup \{-\infty, +\infty\}$, then the following inequality holds for all $0 < p < \infty$:*

$$\mu\big(\{x \in X : |f(x)| \geqslant a\}\big) \leqslant \frac{1}{a^p} \int_X |f|^p \,\mathrm{d}\mu .$$

Theorem A.6 (Layer Cake Representation, [7, Theorem 1.13])**.** *Let ν be a Borel measure on the real line such that*

$$\phi(t) := \nu\big([0, t)\big) < \infty ,$$

for all $t > 0$. This way, $\phi(0) = 0$, and ϕ is monotone nondecreasing, hence measurable.

Let (Ω, Σ, μ) be a measure space and let f be a nonnegative, measurable function on Ω. It then holds

$$\int_\Omega \phi(f(x)) \; \mu(\mathrm{d}x) = \int_0^\infty \mu(\{x \in \Omega \;:\; f(x) > t\}) \; \nu(\mathrm{d}t) .$$

For $\nu(\mathrm{d}t) = pt^{p-1}\,\mathrm{d}t$, we have in particular that

$$\int_\Omega f^p \; \mu(\mathrm{d}x) = \int_0^\infty pt^{p-1} \mu(\{x \in \Omega \;:\; f(x) > t\}) \;\mathrm{d}t .$$

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