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The Moser-Trudinger-Onofri Inequality*

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(*In Honor of the Scientific Contributions of Professor Luc Tartar*)

Abstract This paper is devoted to results on the Moser-Trudinger-Onofri inequality, or the Onofri inequality for brevity. In dimension two this inequality plays a role similar to that of the Sobolev inequality in higher dimensions. After justifying this statement by recovering the Onofri inequality through various limiting procedures and after reviewing some known results, the authors state several elementary remarks.

Various new results are also proved in this paper. A proof of the inequality is given by using mass transportation methods (in the radial case), consistently with similar results for Sobolev inequalities. The authors investigate how duality can be used to improve the Onofri inequality, in connection with the logarithmic Hardy-Littlewood-Sobolev inequality. In the framework of fast diffusion equations, it is established that the inequality is an entropy-entropy production inequality, which provides an integral remainder term. Finally, a proof of the inequality based on rigidity methods is given and a related nonlinear flow is introduced.

Keywords Moser-Trudinger-Onofri inequality, Duality, Mass transportation, Fast diffusion equation, Rigidity
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1 Introduction

In this paper, we consider the Moser-Trudinger-Onofri inequality, or the Onofri inequality, for brevity. This inequality takes any of the three following forms, which are all equivalent.

(1) The Euclidean Onofri inequality:

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x \ge \log\Big(\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu\Big)-\int_{\mathbb{R}^2}u\,\mathrm{d}\mu\,. \tag{1.1}$$

Here $\mathrm{d}\mu=\mu(x)\,\mathrm{d}x$ denotes the probability measure defined by $\mu(x)=\frac{1}{\pi}\,(1+|x|^2)^{-2}$, $x\in\mathbb{R}^2$.

(2) The Onofri inequality on the two-dimensional sphere $\mathbb{S}^2$:

$$\frac{1}{4}\int_{\mathbb{S}^2}|\nabla v|^2\,\mathrm{d}\sigma \ge \log\Big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\Big)-\int_{\mathbb{S}^2}v\,\mathrm{d}\sigma\,. \tag{1.2}$$

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Here $\mathrm{d}\sigma$ denotes the uniform probability measure, that is, the measure induced by Lebesgue's measure on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ divided by a 4π factor.

(3) The Onofri inequality on the two-dimensional cylinder $\mathcal{C} = \mathbb{S}^1 \times \mathbb{R}$:

$$\frac{1}{16\,\pi} \int_{\mathcal{C}} |\nabla w|^2\,\mathrm{d}y \geq \log\Big(\int_{\mathcal{C}} \mathrm{e}^w\,\nu\,\mathrm{d}y\Big) - \int_{\mathcal{C}} w\,\nu\,\mathrm{d}y\,. \tag{1.3}$$

Here $y = (\theta, s) \in \mathcal{C} = \mathbb{S}^1 \times \mathbb{R}$ and $\nu(y) = \frac{1}{4\pi}\,(\cosh s)^{-2}$ is a weight.

These three inequalities are equivalent. Indeed, on $\mathbb{S}^2 \subset \mathbb{R}^3$, let us consider the coordinates $(\omega, z) \in \mathbb{R}^2 \times \mathbb{R}$ such that $|\omega|^2 + z^2 = 1$ and $z \in [-1, 1]$. Let $\rho := |\omega|$ and define the stereographic projection $\Sigma : \mathbb{S}^2 \setminus \{\mathrm{N}\} \to \mathbb{R}^2$ by $\Sigma(\omega) = x = \frac{r\,\omega}{\rho}$ and

$$z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{r^2 + 1}\,, \quad \rho = \frac{2\,r}{r^2 + 1}\,.$$

The north pole N corresponds to $z = 1$ (and is formally sent at infinity) while the equator (corresponding to $z = 0$) is sent onto the unit sphere $\mathbb{S}^1 \subset \mathbb{R}^2$. However, on the cylinder $\mathcal{C}$, we can consider the Emden-Fowler transformation using the coordinates $\theta = \frac{x}{|x|} = \omega\rho$ and $s = -\log r = -\log|x|$. The functions u, v and w in (1.1)–(1.2) and (1.3) are then related by

$$u(x) = v(\omega, z) = w(\theta, s)\,.$$

2 A Review of the Literature

(1.2) was established in [58] without a sharp constant, based on the Moser-Trudinger inequality which was itself proved in [68, 58], and in [61] with a sharp constant. For this reason it is sometimes called the Moser-Trudinger-Onofri inequality in the literature. The result of Onofri strongly relies on a paper of Aubin [4], which contains a number of results of existence for inequalities of the Onofri type on manifolds (with unknown optimal constants). Also based on the Moser-Trudinger inequality, one has to mention [62] which connects (1.2) with the Lebedev-Milin inequalities.

Concerning the other equivalent forms of the inequality, we may refer to [37] for (1.3) while it is more or less a standard result that (1.1) is equivalent to (1.2); an important result concerning this last point is the paper of Carlen and Loss [19], which will be considered in more detail in Section 5. Along the same line of thought, one has to mention [8], which also is based on the Funk-Hecke formula for the dual inequality, as was Lieb's work on Hardy-Littlewood-Sobolev inequalities on the sphere (see [53]).

The optimal function can be identified using the associated Euler-Lagrange equation (see [50, Lemma 3.1] which provides details that are somewhat lacking in Onofri's original paper). We may also refer to [38, Theorem 12] for multiplicity results of a slightly more general equation than the one associated with (1.1).

Another strategy can be found in the existing literature. In [45], Ghigi provided a proof based on the Prékopa-Leindler inequality, which is also explained in full detail in the book [47, Chapters 16–18] of Ghoussoub and Moradifam. Let us mention that the book contains much more material and tackles the issues of improved inequalities under additional constraints, a question that was raised in [4] and later studied in [24–25, 46].

Symmetrization, which allows to prove that optimality in (1.1), (1.2) or (1.3) is achieved among functions that are respectively radial (on the Euclidean space), or depend only on the azimuthal angle (the latitude, on the sphere), or only on the coordinate along the axis (of the cylinder), is an essential tool to reduce the complexity of the problem. For brevity, we shall refer to the symmetric case when the function set is reduced to one of the above cases. Symmetrization results are widespread in the mathematical literature, so we shall only quote a few key papers. A standard reference is the paper of [6] and in particular [6, Theorem 2] which is a key result for establishing the Hardy-Littlewood-Sobolev inequalities on the sphere and its limiting case, the logarithmic Hardy-Littlewood-Sobolev inequality. By duality and by considering the optimality case, one gets a symmetry result for the Onofri inequality, which can be found for instance in [19]. It is also standard that the kinetic energy (Dirichlet integral) is decreased by symmetrization (a standard reference in the Euclidean case can be found in [54, Lemma 7.17]; also see [14, p. 154]) and the adaptation to the sphere is straightforward. Historically, this was known much earlier and one can for instance quote [58] (without any justification) and [3, Lemmas 1–2, p. 586]. This is precisely stated in the context of the Onofri inequality on $\mathbb{S}^2$ in [45, Lemma 1], which itself refers to [5, Corollary 3, p. 60] and [51]. A detailed statement can be found in [47, Lemma 17.1.2]. Competing symmetries are another aspect of symmetrization that we will not study in this paper and for which we refer to [19].

In [65], Rubinstein gave a proof of the Onofri inequality that does not use symmetrization/rearrangement arguments. Also see [66] and in particular [66, Corollary 10.12] which contains a reinforced version of the inequality. In [24, Remark (1), p. 217], there is another proof which does not rely on symmetry, based on a result in [49]. Another proof that went rather unnoticed was used in the paper of Fontenas [42]. This approach is based on the so-called Γ_2 or carré du champ method. In the symmetric case, the problem can be reduced to an inequality involving the ultraspherical operator that we will consider in Section 7 (see (3.9)) with $\lambda = 1$. As far as we know, the first observation concerning this equivalent formulation can be found in [9], although no justification of the symmetrization appears in this paper. In a series of recent papers [31–36], two of the authors clarified the link that connects the carré du champ method with rigidity results that can be found in [10] and earlier papers. Even better, their method involves a nonlinear flow which produces a remainder term, which will be considered in Subsection 7.2.

Spherical harmonics play a crucial but hidden role, so we shall not insist on them and refer to [8] and in the symmetric case, to [47, Chapter 16] for further details. As quoted in [47], other variations on the Onofri-Moser-Trudinger inequality were given in [1, 20, 24–25, 41, 57]. The question of dimensions higher than $d = 2$ is an entire topic by itself and one can refer to [8, 13, 60, 29] for some results in this direction. Various extensions of the Moser-Trudinger and Moser-Trudinger-Onofri inequalities have been proposed, which are out of the scope of this paper; let us simply mention [52] as a contribution in this direction and refer the interested reader to the references therein.

In this paper, we will cover neither issues related to conformal invariance, that were central in [61], nor motivations arising from differential geometry. The reader interested in understanding how Onofri inequality is related to the problem of prescribing the Gaussian curvature on $\mathbb{S}^2$ is invited to refer to [23, Section 3] for an introductory survey, and to [24–26] for more details.

Onofri inequality also has important applications, for instance, in chemotaxis (see [44, 16] in the case of the Keller-Segel model).

As a conclusion of this review, we can list the main tools as follows that we have found in the literature:

(T1) Existence by variational methods;

(T2) symmetrization techniques which allow to reduce the problem for (1.1) to radial functions;

(T3) identification of the solutions to the Euler-Lagrange equations (among radially symmetric functions);

(T4) duality with the logarithmic Hardy-Littlewood-Sobolev inequality and study of the logarithmic Hardy-Littlewood-Sobolev inequality based on spherical harmonics and the Funk-Hecke formula;

(T5) convexity methods related to the Prékopa-Leindler inequality;

(T6) Γ_2 or carré du champ methods;

(T7) limiting procedures based on other functional inequalities.

With these tools, we may try to summarize the strategies of proofs that have been developed. The approach of Onofri is based on (T1)–(T3), while (T4)–(T7) have been used in four independent and alternative strategies of proofs. None of them is elementary, in the sense that they rely on fundamental, deep or rather technical intermediate results.

In this paper, we intend to give new methods which, though not elementary, are slightly simpler, or open new lines of thought. They also provide various additional terms which are all improvements. Several of them are based on the use of nonlinear flows, which, as far as we know, have not been really considered up to now, except in [30, 39]. They borrow some key issues from at least one of the above mentioned tools (T1)–(T7) or enlarge the framework.

(1) Limiting procedures based on other functional inequalities rather than Onofri's, as in (T7), will be considered in Subsection 3.1. Six cases are studied, none of them being entirely new, but we thought that it was quite interesting to collect them. They also justify why we claim that "the Onofri inequality plays in dimension two a role similar to that of the Sobolev inequality in higher dimensions". Other preliminary results (linearization, and (T2): Symmetry results) will be considered in Subsections 3.2–3.3.

(2) Section 4 is devoted to a mass transportation approach of Onofri inequality. Because of (T5), it was expected that such a technique would apply, at least formally (see Subsection 4.1). A rigorous proof is established in the symmetric case in Subsection 4.2 and the consistency with a mass transportation approach of Sobolev inequalities is shown in Subsection 4.3. We have not found any result in this direction in the existing literature. (T2) is needed for a rigorous proof.

(3) In Section 5, we will come back to duality methods, and get a first improvement on the standard Onofri inequality based on a simple expansion of a square. This has of course something to do with (T4)–(T5), but Proposition 5.1 is, as far as we know, a new result. We also introduce the super-fast (or logarithmic) diffusion, which has striking properties in relation to Onofri inequality and duality, but we have not been able to obtain an improvement of the inequality as it was done in the case of Sobolev inequalities in [39].

(4) In Section 6, we observe that in dimension $d = 2$, the Onofri inequality is the natural functional inequality associated with the entropy-entropy production method for the fast diffusion equation with exponent $m = \frac{1}{2}$. It is remarkable that no singular limit has to be taken. Moreover, the entropy-entropy production method provides an integral remainder term which is new.

(5) In Section 7, we establish rigidity results. Existence of optimal functions is granted by (T1). Our results are equivalent to those obtained with Γ_2 or carré du champ methods. This

has already been noticed in the literature (but the equivalence of the two methods has never really been emphasized as it should have been). For the sake of simplicity, we start by a proof in the symmetric case in Subsection 7.1. However, our method does not a priori require (T2) and directly provides essential properties for (T3), that is, the uniqueness of the solutions up to conformal invariance (for the critical value of a parameter, which corresponds to the first bifurcation point from the branch of the trivial constant solutions). Not only this point is remarkable, but we are also able to exhibit a nonlinear flow (in Subsection 7.2) which unifies the various approaches and provides a new integral remainder term. Our main results in this perspective are collected in Subsection 7.3.

3 Preliminaries

3.1 Onofri inequality as a limit of various interpolation inequalities

Onofri inequality appears as an endpoint of various families of interpolation inequalities and corresponds to a critical case in dimension $d = 2$, exactly like Sobolev inequality when $d \geq 3$. This is why one can claim that it plays in dimension two a role similar to that of the Sobolev inequality in higher dimensions. Let us give some six examples of such limits, which are probably the easiest way of proving Onofri inequality.

3.1.1 Onofri inequality as a limit of interpolation inequalities on $\mathbb{S}^2$

On the sphere $\mathbb{S}^2$, one can derive the Onofri inequality from a family of interpolation inequalities on $\mathbb{S}^2$. We start from

$$\frac{q-2}{2}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \|f\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \geq \|f\|^2_{\mathrm{L}^q(\mathbb{S}^2)}\,, \tag{3.1}$$

which holds for any $f \in \mathrm{H}^1(\mathbb{S}^2)$ (see [8, 10, 31]). Proceeding as in [8] (see also [37]), we choose $q = 2\,(1+t)$, $f = 1 + \frac{1}{2\,t}\,v$, for any positive t and use (3.1). This gives

$$\Big(\frac{1}{4\,t}\int_{\mathbb{S}^2} |\nabla v|^2\,\mathrm{d}\sigma + 1 + \frac{1}{t}\int_{\mathbb{S}^2} v\,\mathrm{d}\sigma + \frac{1}{4\,t^2}\int_{\mathbb{S}^2} |v|^2\,\mathrm{d}\sigma\Big)^{1+t} \geq \int_{\mathbb{S}^2}\Big|1 + \frac{1}{2\,t}\,v\Big|^{2\,(1+t)}\,\mathrm{d}\sigma\,.$$

By taking the limit $t \to \infty$, we recover (1.2).

3.1.2 Onofri inequality as a limit of Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities:

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leq \mathsf{C}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}\,, \tag{3.2}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $1 < p \leq \frac{d}{d-2}$ if $d \geq 3$, and $1 < p < \infty$ if $d = 2$. Such an inequality holds for any smooth function f with sufficient decay at infinity and by density, for any function $f \in \mathrm{L}^{p+1}(\mathbb{R}^d)$ such that ∇f is square integrable. We shall assume that $\mathsf{C}_{p,d}$ is the best possible constant. In [28], it was established that equality holds in (3.2) if $f = F_p$ with

$$F_p(x) = (1 + |x|^2)^{-\frac{1}{p-1}}\,, \quad \forall\, x \in \mathbb{R}^d\,, \tag{3.3}$$

and that all extremal functions are equal to F_p up to multiplication by a constant, a translation and a scaling. If $d \geq 3$, the limit case $p = \frac{d}{d-2}$ corresponds to Sobolev inequality and one recovers the results of Aubin and Talenti in [3, 67] with $\theta=1$ as follows: The optimal functions for it are, up to scalings, translations and multiplications by a constant, all equal to $F_{\frac{d}{d-2}}(x)=(1+|x|^2)^{-\frac{d-2}{2}}$, and

$$\mathsf{S}_d=(\mathsf{C}_{\frac{d}{d-2},d})^2\,.$$

We can recover the Euclidean Onofri inequality as the limit case $d=2$, $p\to\infty$ in the above family of inequalities in the following way.

Proposition 3.1 (see [30]) *Assume that $u\in\mathcal{D}(\mathbb{R}^2)$ is such that $\int_{\mathbb{R}^2}u\,\mathrm{d}\mu=0$, and let*

$$f_p:=F_p\left(1+\frac{u}{2\,p}\right),$$

where F_p is defined by (3.3)*. Then we have*

$$1\le\lim_{p\to\infty}\mathsf{C}_{p,2}\,\frac{\|\nabla f_p\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)}\,\|f_p\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}}=\frac{\mathrm{e}^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x}}{\int_{\mathbb{R}^2}\mathrm{e}^{\,u}\,\mathrm{d}\mu}\,.$$

We recall that $\mu(x):=\frac{1}{\pi}\,(1+|x|^2)^{-2}$ and $\mathrm{d}\mu(x):=\mu(x)\,\mathrm{d}x$.

Proof For completeness, let us give a short proof. We can rewrite (3.2) as

$$\frac{\int_{\mathbb{R}^2}|f|^{2p}\,\mathrm{d}x}{\int_{\mathbb{R}^2}|F_p|^{2p}\,\mathrm{d}x}\le\left(\frac{\int_{\mathbb{R}^2}|\nabla f|^2\,\mathrm{d}x}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,\mathrm{d}x}\right)^{\frac{p-1}{2}}\frac{\int_{\mathbb{R}^2}|f|^{p+1}\,\mathrm{d}x}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,\mathrm{d}x}$$

and observe that, with $f=f_p$, we have

(i) $\lim_{p\to\infty}\int_{\mathbb{R}^2}|F_p|^{2p}\,\mathrm{d}x=\int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^2}\,\mathrm{d}x=\pi$ and

$$\lim_{p\to\infty}\int_{\mathbb{R}^2}|f_p|^{2p}\,\mathrm{d}x=\int_{\mathbb{R}^2}F_p^{2p}\Big(1+\frac{u}{2p}\Big)^{2p}\,\mathrm{d}x=\int_{\mathbb{R}^2}\frac{\mathrm{e}^u}{(1+|x|^2)^2}\,\mathrm{d}x\,,$$

so that $\frac{\int_{\mathbb{R}^2}|f_p|^{2p}\,\mathrm{d}x}{\int_{\mathbb{R}^2}|F_p|^{2p}\,\mathrm{d}x}$ converges to $\int_{\mathbb{R}^2}\mathrm{e}^{\,u}\,\mathrm{d}\mu$ as $p\to\infty$.

(ii) $\int_{\mathbb{R}^2}|F_p|^{p+1}\,\mathrm{d}x=\frac{(p-1)\pi}{2}$, $\lim_{p\to\infty}\int_{\mathbb{R}^2}|f_p|^{p+1}\,\mathrm{d}x=\infty$, but

$$\lim_{p\to\infty}\frac{\int_{\mathbb{R}^2}|f_p|^{p+1}\,\mathrm{d}x}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,\mathrm{d}x}=1.$$

(iii) Expanding the square and integrating by parts, we find

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla f_p|^2\,\mathrm{d}x&=\frac{1}{4p^2}\int_{\mathbb{R}^2}F_p^2\,|\nabla u|^2\,\mathrm{d}x-\int_{\mathbb{R}^2}\left(1+\frac{u}{2p}\right)^2F_p\,\Delta F_p\,\mathrm{d}x\\&=\frac{1}{4p^2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+\frac{2\pi}{p+1}+o(p^{-2})\,.\end{aligned}$$

Here we have used $\int_{\mathbb{R}^2}|\nabla F_p|^2\,\mathrm{d}x=\frac{2\pi}{p+1}$ and the condition $\int_{\mathbb{R}^2}u\,\mathrm{d}\mu=0$ in order to discard one additional term of the order of p^{-2}. On the other hand, we find

$$\left(\frac{\int_{\mathbb{R}^2}|\nabla f_p|^2\,\mathrm{d}x}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,\mathrm{d}x}\right)^{\frac{p-1}{2}}\sim\left(1+\frac{p+1}{8\,\pi\,p^2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^{\frac{p-1}{2}}\sim\mathrm{e}^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x}$$

as $p\to\infty$. Collecting these estimates concludes the proof.

3.1.3 Onofri inequality as a limit of Sobolev inequalities

Another way to derive Onofri inequality is to consider the usual optimal Sobolev inequalities in $\mathbb{R}^2$, written for an $\mathrm{L}^p(\mathbb{R}^2)$ norm of the gradient, for an arbitrary $p \in (1,2)$. This method is inspired by [29], which is devoted to inequalities in the exponential form in dimensions $d \geq 2$ (see in particular [29, Example 1.2]). In the special case $p \in (1,2)$, $d = 2$, let us consider the Sobolev inequality

$$\|f\|^p_{\mathrm{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)} \leq \mathsf{C}_p \, \|\nabla f\|^p_{\mathrm{L}^p(\mathbb{R}^2)}, \quad \forall \, f \in \mathcal{D}(\mathbb{R}^2)\,, \tag{3.4}$$

where equality is achieved by the Aubin-Talenti extremal profile

$$f_\star(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{2-p}{p}}, \quad \forall \, x \in \mathbb{R}^2\,.$$

The extremal functions were already known from the celebrated papers by Aubin and Talenti [3, 67]. See also [12, 64] for earlier related computations, which provided the value of some of the best constants. It is easy to check that $f_\star$ solves

$$-\Delta_p f_\star = 2\left(\frac{2-p}{p-1}\right)^{p-1} f_\star^{\frac{2p}{2-p}-1}\,,$$

and hence

$$\|\nabla f_\star\|^p_{\mathrm{L}^p(\mathbb{R}^2)} = \frac{1}{\mathsf{C}_p}\, \|f_\star\|^p_{\mathrm{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)} = 2\left(\frac{2-p}{p-1}\right)^{p-1} \|f_\star\|^{\frac{2p}{2-p}}_{\mathrm{L}^{\frac{2p}{2-p}}(\mathbb{R}^2)}\,,$$

so that the optimal constant is

$$\mathsf{C}_p = \frac{1}{2}\left(\frac{p-1}{2-p}\right)^{p-1} \left(\frac{p^2 \left|\sin\left(\frac{2\pi}{p}\right)\right|}{2\,(p-1)\,(2-p)\,\pi^2}\right)^{\frac{p}{2}}.$$

We can study the limit $p \to 2_-$ in order to recover the Onofri inequality by considering $f = f_\star\left(1 + \frac{2-p}{2\,p}\, u\right)$, where u is a given smooth, compactly supported function, and $\varepsilon = \frac{2-p}{2\,p}$. A direct computation gives

$$\lim_{p\to 2_-} \int_{\mathbb{R}^2} f^{\frac{2p}{2-p}}\, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{\mathrm{e}^u}{(1+|x|^2)^2}\, \mathrm{d}x = \pi \int_{\mathbb{R}^2} \mathrm{e}^u\, \mathrm{d}\mu$$

and

$$\int_{\mathbb{R}^2} |\nabla f|^p\, \mathrm{d}x = 2\,\pi\,(2-p)\left[1 + \frac{2-p}{2}\int_{\mathbb{R}^2} u\, \mathrm{d}\mu\right] + \left(\frac{2-p}{2\,p}\right)^p \int_{\mathbb{R}^2} |\nabla u|^2\, \mathrm{d}x + o((2-p)^2)$$

as $p \to 2_-$. By taking the logarithm of both sides of (3.4), we get

$$\begin{aligned}
\frac{2-p}{2}\, \log\left(\int_{\mathbb{R}^2} \mathrm{e}^u\, \mathrm{d}\mu\right) &\sim \frac{2-p}{2}\, \log\left(\frac{\int_{\mathbb{R}^2} f^{\frac{2p}{2-p}}\, \mathrm{d}x}{\int_{\mathbb{R}^2} f_\star^{\frac{2p}{2-p}}\, \mathrm{d}x}\right)\\
&\leq \log\left(\frac{\int_{\mathbb{R}^2} |\nabla f|^p\, \mathrm{d}x}{\int_{\mathbb{R}^2} |\nabla f_\star|^p\, \mathrm{d}x}\right)\\
&= \log\left(1 + \frac{2-p}{2}\int_{\mathbb{R}^2} u\, \mathrm{d}\mu + \frac{2-p}{32\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\, \mathrm{d}x + o(2-p)\right).
\end{aligned}$$

Gathering the terms of order $2-p$, we recover the Euclidean Onofri inequality by passing to the limit $p \to 2_-$.

3.1.4 The radial Onofri inequality as a limit when $d \to 2$

Although this approach is restricted to radially symmetric functions, one of the most striking ways to justify the fact that the Onofri inequality plays in dimension two a role similar to that of the Sobolev inequality in higher dimensions goes as follows. To start with, one can consider the Sobolev inequality applied to radially symmetric functions only. The dimension d can now be considered as a real parameter. Then, by taking the limit $d \to 2$, one can recover a weaker (i.e. for radial functions only) version of the Onofri inequality. The details of the computation, taken from [39], follow.

Consider the radial Sobolev inequality

$$\mathsf{s}_d \int_0^\infty |f'|^2\, r^{d-1}\, \mathrm{d}r \ge \Big(\int_0^\infty |f|^{\frac{2\,d}{d-2}}\, r^{d-1}\, \mathrm{d}r \Big)^{1-\frac{2}{d}} \tag{3.5}$$

with an optimal constant

$$\mathsf{s}_d = \frac{4}{d\,(d-2)} \Big(\frac{\Gamma\big(\frac{d+1}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{d}{2}\big)} \Big)^{\frac{2}{d}}\,.$$

We may pass to the limit in (3.5) with the choice

$$f(r) = f_\star(r) \Big(1 + \frac{d-2}{2\,d}\, u \Big)\,,$$

where $f_\star(r) = (1+r^2)^{-\frac{d-2}{2}}$ gives the equality case in (3.5), to get the radial version of Onofri inequality for u. By expanding the expression of $|f'|^2$, we get

$$f'^2 = f_\star'^2 + \frac{d-2}{d}\, f_\star'(f_\star\, u)' + \Big(\frac{d-2}{2\,d}\Big)^2 (f_\star'\, u + f_\star\, u')^2\,.$$

We have

$$\lim_{d\to 2_+} \int_0^\infty \Big| f_\star \Big(1 + \frac{d-2}{2\,d}\, u\Big)\Big|^{\frac{2\,d}{d-2}}\, r^{d-1}\, \mathrm{d}r = \int_0^\infty \mathrm{e}^u\, \frac{r\,\mathrm{d}r}{(1+r^2)^2}\,,$$

so that, as $d \to 2_+$,

$$\Big(\int_0^\infty \Big| f_\star \Big(1 + \frac{d-2}{2\,d}\, u\Big)\Big|^{\frac{2\,d}{d-2}}\, r^{d-1}\, \mathrm{d}r \Big)^{\frac{d-2}{d}} - 1 \sim \frac{d-2}{2}\, \log \Big(\int_0^\infty \mathrm{e}^u\, \frac{r\,\mathrm{d}r}{(1+r^2)^2} \Big).$$

Also, using the fact that

$$\mathsf{s}_d = \frac{1}{d-2} + \frac{1}{2} - \frac{1}{2}\, \log 2 + o(1) \quad \text{as } d \to 2_+\,,$$

we have

$$\mathsf{s}_d \int_0^\infty |f'|^2\, r^{d-1}\, \mathrm{d}r \sim 1 + (d-2) \Big[\frac{1}{8} \int_0^\infty |u'|^2\, r\, \mathrm{d}r + \int_0^\infty u\, \frac{2\,r\,\mathrm{d}r}{(1+r^2)^2} \Big]\,.$$

By keeping only the highest-order terms, which are of the order of $(d-2)$, and passing to the limit as $d \to 2_+$ in (3.5), we obtain

$$\frac{1}{8} \int_0^\infty |u'|^2\, r\, \mathrm{d}r + \int_0^\infty u\, \frac{2\,r\,\mathrm{d}r}{(1+r^2)^2} \ge \log \Big(\int_0^\infty \mathrm{e}^u\, \frac{2\,r\,\mathrm{d}r}{(1+r^2)^2} \Big)\,,$$

which is Onofri inequality written for radial functions.

3.1.5 Onofri inequality as a limit of Caffarelli-Kohn-Nirenberg inequalities

Onofri inequality can be obtained as the limit in a familly of Caffarelli-Kohn-Nirenberg inequalities, as was first done in [37].

Let $2^* := \infty$ if $d = 1$ or 2, and $2^* := 2\,\frac{d}{d-2}$ if $d \geq 3$ and $a_c := \frac{d-2}{2}$. Consider the space $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$ with respect to the norm $u \mapsto \|\,|x|^{-a}\,\nabla u\,\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. In this section, we shall consider the Caffarelli-Kohn-Nirenberg inequalities

$$\Big(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,\mathrm{d}x\Big)^{\frac{2}{p}} \leq \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,\mathrm{d}x\;. \tag{3.6}$$

These inequalities generalize to $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ the Sobolev inequality, and in particular, the exponent p is given in terms of a and b by

$$p = \frac{2\,d}{d-2+2\,(b-a)}\;,$$

as can be checked by a simple scaling argument. A precise statement on the range of validity of (3.6) goes as follows.

Lemma 3.1 (see [15]) *Let $d \geq 1$. For any $p \in [2, 2^*]$ if $d \geq 3$, or $p \in [2, 2^*)$ if $d = 1$ or 2, there exists a positive constant $\mathsf{C}_{a,b}$ such that (3.6) holds if a, b and p are related by $b = a - a_c + \frac{d}{p}$, with the restrictions $a < a_c$, $a \leq b \leq a+1$ if $d \geq 3$, $a < b \leq a+1$ if $d = 2$ and $a + \frac{1}{2} < b \leq a+1$ if $d = 1$.*

We shall restrict our purpose to the case of dimension $d = 2$. For any $\alpha \in (-1, 0)$, let us denote by $\mathrm{d}\mu_\alpha$ the probability measure on $\mathbb{R}^2$ defined by $\mathrm{d}\mu_\alpha := \mu_\alpha\,\mathrm{d}x$ where

$$\mu_\alpha := \frac{1+\alpha}{\pi}\,\frac{|x|^{2\,\alpha}}{(1+|x|^{2\,(1+\alpha)})^2}\;.$$

It was established in [37] that

$$\log\Big(\int_{\mathbb{R}^2} \mathrm{e}^{\,u}\,\mathrm{d}\mu_\alpha\Big) - \int_{\mathbb{R}^2} u\,\mathrm{d}\mu_\alpha \leq \frac{1}{16\,\pi\,(1+\alpha)} \int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x, \quad \forall\, u \in \mathcal{D}(\mathbb{R}^2), \tag{3.7}$$

where $\mathcal{D}(\mathbb{R}^2)$ is the space of smooth functions with compact support. By density with respect to the natural norm defined by each of the inequalities, the result also holds on the corresponding Orlicz space.

We adopt the strategy of [37, Subsection 2.3] to pass to the limit in (3.6) as $(a,b) \to (0,0)$ with $b = \frac{\alpha}{\alpha+1}\,a$. Let

$$a_\varepsilon = -\frac{\varepsilon}{1-\varepsilon}\,(\alpha+1)\,, \quad b_\varepsilon = a_\varepsilon + \varepsilon, \quad p_\varepsilon = \frac{2}{\varepsilon}$$

and

$$u_\varepsilon(x) = (1+|x|^{2\,(\alpha+1)})^{-\frac{\varepsilon}{1-\varepsilon}}\;.$$

Assuming that u_ε is an optimal function for (3.6), we define

$$\kappa_\varepsilon = \int_{\mathbb{R}^2} \Big[\frac{u_\varepsilon}{|x|^{a_\varepsilon+\varepsilon}}\Big]^{\frac{2}{\varepsilon}} \mathrm{d}x = \int_{\mathbb{R}^2} \frac{|x|^{2\,\alpha}}{(1+|x|^{2\,(1+\alpha)})^2}\,\frac{u_\varepsilon^2}{|x|^{2a_\varepsilon}}\,\mathrm{d}x = \frac{\pi}{\alpha+1}\,\frac{\Gamma\big(\frac{1}{1-\varepsilon}\big)^2}{\Gamma\big(\frac{2}{1-\varepsilon}\big)}\,,$$

$$\lambda_\varepsilon = \int_{\mathbb{R}^2} \Big[\frac{|\nabla u_\varepsilon|}{|x|^a}\Big]^2 \mathrm{d}x = 4\,a_\varepsilon^2 \int_{\mathbb{R}^2} \frac{|x|^{2\,(2\,\alpha+1-a_\varepsilon)}}{(1+|x|^{2\,(1+\alpha)})^{\frac{2}{1-\varepsilon}}}\,\mathrm{d}x = 4\,\pi\,\frac{|a_\varepsilon|}{1-\varepsilon}\,\frac{\Gamma\big(\frac{1}{1-\varepsilon}\big)^2}{\Gamma\big(\frac{2}{1-\varepsilon}\big)}\;.$$

Then $w_\varepsilon = (1 + \frac{1}{2}\,\varepsilon\,u)\,u_\varepsilon$ is such that

$$\lim_{\varepsilon\to0_+} \frac{1}{\kappa_\varepsilon} \int_{\mathbb{R}^2} \frac{|w_\varepsilon|^{p_\varepsilon}}{|x|^{b_\varepsilon p_\varepsilon}}\,\mathrm{d}x = \int_{\mathbb{R}^2} \mathrm{e}^u\,\mathrm{d}\mu_\alpha\,,$$
$$\lim_{\varepsilon\to0_+} \frac{1}{\varepsilon}\Big[\frac{1}{\lambda_\varepsilon} \int_{\mathbb{R}^2} \frac{|\nabla w_\varepsilon|^2}{|x|^{2a_\varepsilon}}\,\mathrm{d}x - 1\Big] = \int_{\mathbb{R}^2} u\,\mathrm{d}\mu_\alpha + \frac{1}{16\,(1+\alpha)\,\pi}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)}\,.$$

3.1.6 Limits of some Gagliardo-Nirenberg inequalities on the line

Onofri inequality on the cylinder, (1.3), can also be recovered by a limiting process, in the symmetric case. As far as we know, this method for proving the inequality is new, but a clever use of the Emden-Fowler transformation and of the results based on the Caffarelli-Kohn-Nirenberg inequalities shows that this is to be expected (see [37] for more considerations in this direction).

Consider the Gagliardo-Nirenberg inequalities on the line

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}^p\,\|f'\|^\theta_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R})}, \quad \forall\,f\in\mathrm{H}^1(\mathbb{R})$$

with $\theta = \frac{p-2}{2\,p}$, $p>2$. The equality is achieved by the function

$$f_\star(x) := (\cosh s)^{-\frac{2}{p-1}}, \quad \forall\,s\in\mathbb{R}$$

(see [34] for details). By taking the logarithm of both sides of the inequality, we find that

$$\frac{2}{p}\,\log\Big(\frac{\int_{\mathbb{R}} f^p\,\mathrm{d}s}{\int_{\mathbb{R}} f_\star^p\,\mathrm{d}s}\Big) \le \theta\,\log\Big(\frac{\int_{\mathbb{R}} |f'|^2\,\mathrm{d}s}{\int_{\mathbb{R}} |f_\star'|^2\,\mathrm{d}s}\Big) + (1-\theta)\,\log\Big(\frac{\int_{\mathbb{R}} f^2\,\mathrm{d}s}{\int_{\mathbb{R}} f_\star^2\,\mathrm{d}s}\Big)$$

and elementary computations show that as $p\to+\infty$, $f_\star^p \to 2\,\xi$ and $-f_\star\,f_\star'' \sim \frac{4}{p}\,\xi$ with $\xi(s) := \frac{1}{2}\,(\cosh s)^{-2}$. If we take $f = f_\star\big(1+\frac{w}{p}\big)$, we have

$$\lim_{p\to\infty} \int_{\mathbb{R}} f^p\,\mathrm{d}s = 2\int_{\mathbb{R}} \mathrm{e}^w\,\xi\,\mathrm{d}s\,,$$
$$\lim_{p\to\infty} \log\Big(\frac{\int_{\mathbb{R}} f^p\,\mathrm{d}s}{\int_{\mathbb{R}} f_\star^p\,\mathrm{d}s}\Big) = \log\Big(\int_{\mathbb{R}} \mathrm{e}^w\,\xi\,\mathrm{d}s\Big)\,.$$

We can also compute

$$\int_{\mathbb{R}} |f_\star|^2\,\mathrm{d}s = \frac{p-1}{2} + 2\,\log 2 + O\Big(\frac{1}{p}\Big),$$
$$\int_{\mathbb{R}} |f|^2\,\mathrm{d}s = \int_{\mathbb{R}} |f_\star|^2\,\Big(1+\frac{2}{p}\,w + \frac{1}{p^2}\,w^2\Big)\,\mathrm{d}s = \frac{p-1}{2} + 2\,\log 2 + O\Big(\frac{1}{p}\Big)$$

as $p\to+\infty$, so that

$$\frac{\int_{\mathbb{R}} f^2\,\mathrm{d}s}{\int_{\mathbb{R}} f_\star^2\,\mathrm{d}s} - 1 = O\Big(\frac{1}{p^2}\Big) \quad \text{and} \quad \lim_{p\to\infty} p\,\log\Big(\frac{\int_{\mathbb{R}} f^2\,\mathrm{d}s}{\int_{\mathbb{R}} f_\star^2\,\mathrm{d}s}\Big) = 0\,.$$

For the last term, we observe that, pointwise,

$$-f_\star\,f_\star'' \sim \frac{2}{p}\,\frac{1}{(\cosh s)^2}$$

and

$$\int_{\mathbb{R}} |f_\star'|^2\,\mathrm{d}s = -\int_{\mathbb{R}} f_\star\, f_\star''\,\mathrm{d}s = \frac{2}{p} + O\Big(\frac{1}{p^2}\Big) \quad \text{as } p \to +\infty\,.$$

Passing to the limit as $p \to +\infty$, we get that

$$\begin{aligned}\int_{\mathbb{R}} |f'|^2\,\mathrm{d}s &= \frac{1}{p^2}\int_{\mathbb{R}} f_\star^2\,|w'|^2\,\mathrm{d}s - \int_{\mathbb{R}} f_\star\, f_\star''\,\Big(1+\frac{w}{p}\Big)^2\,\mathrm{d}s\\ &= \frac{1}{p^2}\int_{\mathbb{R}} |w'|^2\,\mathrm{d}s + \frac{2}{p}\Big(1+\frac{4}{p}\int_{\mathbb{R}} w\,\xi\,\mathrm{d}s\Big) + o\Big(\frac{1}{p^2}\Big)\,,\end{aligned}$$

and finally

$$\log\Big(\frac{\int_{\mathbb{R}}|f'|^2\,\mathrm{d}s}{\int_{\mathbb{R}}|f_\star'|^2\,\mathrm{d}s}\Big) \sim \frac{1}{p}\Big(4\int_{\mathbb{R}} w\,\xi\,\mathrm{d}s + \frac{1}{2}\int_{\mathbb{R}}|w'|^2\,\mathrm{d}s\Big) + o\Big(\frac{1}{p}\Big)\,.$$

Collecting terms, we find that

$$\frac{1}{8}\int_{\mathbb{R}} |w'|^2\,\mathrm{d}s \ge \log\Big(\int_{\mathbb{R}} \mathrm{e}^w\,\xi\,\mathrm{d}s\Big) - \int_{\mathbb{R}} w\,\xi\,\mathrm{d}s\,.$$

3.2 Linearization and the optimal constant

Consider (1.2) and define

$$\mathcal{I}_\lambda := \inf_{\substack{v\in\mathrm{H}^1(\mathbb{S}^2)\\ \int_{\mathbb{S}^2} v\,\mathrm{d}\sigma>0}} \mathcal{Q}_\lambda[v] \quad \text{with} \quad \mathcal{Q}_\lambda[v] := \frac{\frac{1}{4}\int_{\mathbb{S}^2}|\nabla v|^2\,\mathrm{d}\sigma + \lambda\int_{\mathbb{S}^2} v\,\mathrm{d}\sigma}{\log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big)}\,.$$

By Jensen's inequality, $\log(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma) \ge \int_{\mathbb{S}^2} v\,\mathrm{d}\sigma > 0$, so that $\mathcal{I}_\lambda$ is well defined and nonnegative for any $\lambda > 0$. Since constant functions are admissible, we also know that

$$\mathcal{I}_\lambda \le \lambda$$

for any $\lambda > 0$. Moreover, since $\lambda \mapsto \mathcal{Q}_\lambda[v]$ is affine, we know that $\lambda \mapsto \mathcal{I}_\lambda$ is concave and continuous. Assume now that $\int_{\mathbb{S}^2} v\,\mathrm{d}\sigma = 0$ and for any $c > 0$, and let us consider

$$\mathcal{Q}_\lambda[v+c] = \frac{\frac{1}{4}\int_{\mathbb{S}^2}|\nabla v|^2\,\mathrm{d}\sigma + \lambda\,c}{c + \log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big)} \ge \frac{\log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big) + \lambda\,c}{c + \log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big)}\,, \tag{3.8}$$

where the inequality follows from (1.2). It is clear that for such functions v,

$$\begin{aligned}&\lim_{c\to+\infty}\mathcal{Q}_\lambda[v+c] = \lambda\,,\\ &\lim_{c\to 0_+}\mathcal{Q}_\lambda[v+c] = \frac{\int_{\mathbb{S}^2}|\nabla v|^2\,\mathrm{d}\sigma}{\log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big)} = \mathcal{Q}_\lambda[v]\,.\end{aligned}$$

If $\lambda < 1$, using (3.8), we can write that for all $c > 0$,

$$\mathcal{Q}_\lambda[v+c] \ge \lambda + (1-\lambda)\,\frac{\log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big)}{c + \log\big(\int_{\mathbb{S}^2}\mathrm{e}^v\,\mathrm{d}\sigma\big)} \ge \lambda\,,$$

thus proving that $\mathcal{I}_\lambda = \lambda$ is optimal when $\lambda < 1$.

When $\lambda \ge 1$, we may take $v = \varepsilon\,\phi$, where ϕ is an eigenfunction of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^2}$ on the sphere $\mathbb{S}^2$, such that $-\Delta_{\mathbb{S}^2}\phi = 2\,\phi$ and take the limit as $\varepsilon \to 0_+$, so that

$\int_{\mathbb{S}^2} |\nabla v|^2 \,d\sigma = \varepsilon^2 \int_{\mathbb{S}^2} |\nabla \phi|^2 \,d\sigma = 2\,\varepsilon^2 \int_{\mathbb{S}^2} |\phi|^2 \,d\sigma$ and $\log(\int_{\mathbb{S}^2} e^v \,d\sigma) = \log\big(1 + \frac{1}{2}\,\varepsilon^2 \int_{\mathbb{S}^2} \phi^2 \,d\sigma + o(\varepsilon^2)\big)$. Collecting terms, we get that

$$\lim_{\varepsilon\to 0_+} \mathcal{Q}_\lambda[\varepsilon\,\phi] = 1\,.$$

Altogether, we have found that

$$\mathcal{I}_\lambda = \min\{\lambda, 1\}, \quad \forall\,\lambda > 0\,.$$

3.3 Symmetrization results

For the sake of completeness, let us state a result of symmetry. Consider the functional

$$\mathcal{G}_\lambda[v] := \frac{1}{4}\int_{\mathbb{S}^2} |\nabla v|^2 \,d\sigma + \lambda \int_{\mathbb{S}^2} v \,d\sigma - \log\Big(\int_{\mathbb{S}^2} e^v \,d\sigma\Big)\,,$$

and denote by $\mathrm{H}^1_*(\mathbb{S}^2)$ the function in $\mathrm{H}^1(\mathbb{S}^2)$ which depends only on the azimuthal angle (latitude), denoted by $\theta \in [0,\pi]$.

Proposition 3.2 *For any $\lambda > 0$,*

$$\inf_{v\in \mathrm{H}^1(\mathbb{S}^2)} \mathcal{G}_\lambda[v] = \inf_{v\in \mathrm{H}^1_*(\mathbb{S}^2)} \mathcal{G}_\lambda[v]\,.$$

We refer to [47, Lemma 17.1.2] for a proof of the symmetry result and to Section 2 for further historical references.

Hence, for any function $v \in \mathrm{H}^1(\mathbb{S}^2)$, the inequality $\mathcal{G}_1[v] \ge 0$ reads

$$\frac{1}{8}\int_0^\pi |v'(\theta)|^2 \,\sin\theta \,d\theta + \frac{1}{2}\int_0^\pi v(\theta)\,\sin\theta\,d\theta \ge \log\Big(\frac{1}{2}\int_0^\pi e^v\,\sin\theta\,d\theta\Big)\,.$$

The change of variables $z = \cos\theta$ and $v(\theta) = f(z)$ allows to rewrite this inequality as

$$\frac{1}{8}\int_{-1}^1 |f'|^2\,\nu\,dz + \frac{1}{2}\int_{-1}^1 f\,dz \ge \log\Big(\frac{1}{2}\int_{-1}^1 e^f\,dz\Big)\,, \tag{3.9}$$

where $\nu(x) := 1 - z^2$. Let us define the ultraspherical operator $\mathcal{L}$ by $\langle f_1, \mathcal{L}\,f_2\rangle = -\int_{-1}^1 f_1'\,f_2'\,\nu\,dz$ where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathrm{L}^2(-1,1;dz)$. Explicitly we have

$$\mathcal{L}\,f := (1-z^2)\,f'' - 2\,z\,f' = \nu\,f'' + \nu'\,f'$$

and (3.9) simply means

$$-\frac{1}{8}\langle f, \mathcal{L}\,f\rangle + \frac{1}{2}\int_{-1}^1 f\,\nu\,dz \ge \log\Big(\frac{1}{2}\int_{-1}^1 e^f\,\nu\,dz\Big)\,.$$

4 Mass Transportation

Since Onofri inequality appears as a limit case of Sobolev inequalities which can be proved by mass transportation according to [27], it makes a lot of sense to look for a proof based on such techniques. Let us start by recalling some known results.

Assume that F and G are two probability distributions on $\mathbb{R}^2$ and consider the convex function ϕ such that G is the push-forward of F through $\nabla\phi$

$$\nabla\phi_* F = G\,,$$

where $\nabla\phi$ is the Brenier map and ϕ solves the Monge-Ampère equation

$$F = G(\nabla\phi)\det(\mathrm{Hess}(\phi)) \quad \text{on } \mathbb{R}^d \tag{4.1}$$

(see [55] for details). Here $d = 2$, but to emphasize the role of the dimension, we will keep it as a parameter for a while. The Monge-Ampère equation (4.1) holds in the $F\,\mathrm{d}x$ sense almost everywhere according to [56, Remark 4.5], as discussed in [27]. By now this strategy is standard and we shall refer to [69] for background material and technical issues that will be omitted here. We can, for instance, assume that F and G are smooth functions and argue by density afterwards.

4.1 A formal approach

Let us start with a formal computation. Using (4.1), since

$$G(\nabla\phi)^{-\frac{1}{d}} = F^{-\frac{1}{d}}\det(\mathrm{Hess}(\phi))^{\frac{1}{d}} \le \frac{1}{d}\,F^{-\frac{1}{d}}\,\Delta\phi$$

by the arithmetic-geometric inequality, we get the estimate

$$\int_{\mathbb{R}^d} G(y)^{1-\frac{1}{d}}\,\mathrm{d}y = \int_{\mathbb{R}^d} G(\nabla\phi)^{1-\frac{1}{d}}\det(\mathrm{Hess}(\phi))\,\mathrm{d}x \le \frac{1}{d}\int_{\mathbb{R}^d} F^{1-\frac{1}{d}}(x)\,\Delta\phi\,\mathrm{d}x \tag{4.2}$$

using the change of variables $y = \nabla\phi(x)$ and (4.1). Assume that

$$G(y) = \mu(y) = \frac{1}{\pi\,(1+|y|^2)^2}\,, \quad \forall\, y \in \mathbb{R}^d$$

and

$$F = \mu\,\mathrm{e}^u\,.$$

With $d = 2$, we obtain

$$\begin{aligned} 4\int_{\mathbb{R}^2}\sqrt{\mu}\,\mathrm{d}x = 2\,d\int_{\mathbb{R}^d} G(y)^{1-\frac{1}{d}}\,\mathrm{d}y &= 2\int_{\mathbb{R}^d} F^{1-\frac{1}{d}}(x)\,\Delta\phi\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^2}\nabla\log F\cdot\sqrt{F}\,\nabla\phi\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^2}(\nabla\log\mu+\nabla u)\cdot\sqrt{F}\,\nabla\phi\,\mathrm{d}x\,, \end{aligned}$$

which can be estimated using the Cauchy-Schwarz inequality by

$$\begin{aligned} 16\Big(\int_{\mathbb{R}^2}\sqrt{\mu}\,\mathrm{d}x\Big)^2 &= \Big(\int_{\mathbb{R}^2}(\nabla\log\mu+\nabla u)\cdot\sqrt{F}\,\nabla\phi\,\mathrm{d}x\Big)^2 \\ &\le \int_{\mathbb{R}^2}|\nabla u+\nabla\log\mu|^2\,\mathrm{d}x\int_{\mathbb{R}^2} F\,|\nabla\phi|^2\,\mathrm{d}x\,. \end{aligned}$$

If we expand the square, that is, if we write

$$\int_{\mathbb{R}^2}|\nabla u+\nabla\log\mu|^2\,\mathrm{d}x = \int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x - 2\int_{\mathbb{R}^2} u\,\Delta\log\mu\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla\log\mu|^2\,\mathrm{d}x\,,$$

after recalling that

$$-\Delta \log \mu = 8\,\pi\,\mu\,,$$

and after undoing the change of variables $y = \nabla\phi(x)$, so that we get

$$\int_{\mathbb{R}^2} F\,|\nabla\phi|^2\,\mathrm{d}x = \int_{\mathbb{R}^2} G(y)\,|y|^2\,\mathrm{d}x = \int_{\mathbb{R}^2} \mu\,|y|^2\,\mathrm{d}x\,,$$

we end up, after collecting the terms, with

$$\frac{16\big(\int_{\mathbb{R}^2}\sqrt{\mu}\,\mathrm{d}x\big)^2}{\int_{\mathbb{R}^2}\mu\,|y|^2\,\mathrm{d}x} - \int_{\mathbb{R}^2}|\nabla\log\mu|^2\,\mathrm{d}x \le \int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x + 16\,\pi\int_{\mathbb{R}^2} u\,\mathrm{d}\mu\,.$$

Still on a formal level, we may observe that

$$\begin{aligned}16\Big(\int_{\mathbb{R}^2}\sqrt{\mu}\,\mathrm{d}x\Big)^2 &= \Big(-\,2\int_{\mathbb{R}^2} y\cdot\nabla\sqrt{\mu}\,\mathrm{d}x\Big)^2 = \Big(\int_{\mathbb{R}^2} y\,\sqrt{\mu}\cdot\nabla\log\mu\,\mathrm{d}x\Big)^2\\ &\le \int_{\mathbb{R}^2}\mu\,|y|^2\,\mathrm{d}x\int_{\mathbb{R}^2}|\nabla\log\mu|^2\,\mathrm{d}x\end{aligned}$$

as it can easily be checked that $y\sqrt{\mu}$ and $\nabla\log\mu$ are proportional. This would prove Onofri inequality since $\log(\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu) = \log\big(\int_{\mathbb{R}^2}F\,\mathrm{d}x\big) = 0$, if $y\mapsto\sqrt{\mu}$, $y\mapsto\mu\,|y|^2$ and $y\mapsto|\nabla\log\mu|^2$ were integrable, but this is not the case. As we shall see in the next section, this issue can be solved by working on balls.

4.2 The radially symmetric case

When F and G are assumed to depend only on $r=|x|$, so that we may write $|y| = s = \varphi(r)$, and then (4.1) becomes

$$(G\circ\varphi')\,\Big(\frac{\varphi'}{r}\Big)^{d-1}\varphi'' = F$$

which allows to compute φ' using

$$\int_0^{\varphi'(R)} G(s)\,s^{d-1}\,\mathrm{d}s = \int_0^R (G\circ\varphi')\,\Big(\frac{\varphi'}{r}\Big)^{d-1}\varphi''\,r^{d-1}\,\mathrm{d}r = \int_0^R F(r)\,r^{d-1}\,\mathrm{d}r\,.$$

With a straightforward abuse of notation, we shall indifferently write that F is a function of x or of r and G a function of y or s.

The proof is similar to the one in Subsection 4.1 except that all integrals can be restricted to a ball B_R of radius $R>0$ with its center at the origin. Assume that $G = \frac{\mu}{Z_R}$ and $F = \mathrm{e}^u\,\frac{\mu}{Z_R}$ where $Z_R = \int_{B_R}\mu\,\mathrm{d}x$ and u has compact support inside the ball B_R. An easy computation shows that

$$Z_R = \frac{R^2}{1+R^2}\,,\quad\forall\,R>0\,.$$

We shall also assume that u is normalized so that $\int_{B_R}F\,\mathrm{d}x = 1$.

All computations are now done on B_R. The only differences to Subsection 4.1 arise from the integrations by parts, so we have to handle two additional terms as follows:

$$\begin{aligned}&\int_{B_R} F^{1-\frac{1}{d}}(x)\,\Delta\phi\,\mathrm{d}x + \frac{1}{2}\int_{B_R}\nabla\log F\cdot\sqrt{F}\,\nabla\phi\,\mathrm{d}x\\ &= \pi\,R\,\sqrt{F(R)}\,\varphi'(R) = \pi\,R\,\sqrt{\frac{\mu(R)}{Z_R}}\,\varphi'(R)\end{aligned}$$

and

$$2\int_{B_R}\nabla u\cdot\nabla\log\mu\,\mathrm{d}x+2\int_{B_R}u\,\Delta\log\mu\,\mathrm{d}x=4\,\pi\,R\,(\log\mu)'(R)\,u(R)=0\,.$$

If we fix u (smooth, with compact support) and let $R\to\infty$, then it is clear that none of these two terms plays a role. Notice that there exists a constant κ such that

$$\frac{(\varphi'(R))^2}{1+(\varphi'(R))^2}=\frac{R^2}{1+R^2}+\kappa$$

for large values of R, and hence $\varphi'(R)\sim R$. Hence,

$$\lim_{R\to\infty}\pi\,R\,\sqrt{\frac{\mu(R)}{Z_R}}\,\varphi'(R)=\sqrt{\pi}\,.$$

After collecting the terms, we obtain

$$\begin{aligned}&\frac{16\left(\int_{B_R}\sqrt{\mu}\,\mathrm{d}y-\sqrt{\pi}\right)^2}{\int_{B_R}\mu\,|y|^2\,\mathrm{d}y}-\int_{B_R}|\nabla\log\mu|^2\,\mathrm{d}y+o(1)\\&\le\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x-16\,\pi\int_{\mathbb{R}^2}u\,\mathrm{d}\mu\end{aligned}$$

as $R\to\infty$. Using the equality case for the Cauchy-Schwarz inequality once more, we have

$$16\Big(\int_{B_R}\sqrt{\mu}\,\mathrm{d}y-\sqrt{\pi}\Big)^2=\Big(-\,2\int_{B_R}y\cdot\nabla\sqrt{\mu}\,\mathrm{d}y\Big)^2\le\int_{B_R}\mu\,|y|^2\,\mathrm{d}y\int_{B_R}|\nabla\log\mu|^2\,\mathrm{d}y\,.$$

This establishes the result in the radial case.

4.3 Mass transportation for approximating critical Sobolev inequalities

Inspired by the limit of Subsection 3.1.3, we can indeed obtain Onofri inequality as a limiting process of critical Sobolev inequalities involving mass transportation. Let us recall the method of [27]. Let us consider the case where $p<d=2$,

$$F=f^{\frac{d\,p}{d-p}}\,,$$

G are two probability measures, $p'=\frac{p}{p-1}$ is the Hölder conjugate exponent of p and consider the critical Sobolev inequality

$$\|f\|^p_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^d)}\le\mathsf{C}_{p,d}\,\|\nabla f\|^p_{\mathrm{L}^p(\mathbb{R}^d)}\,,\quad\forall\,f\in\mathcal{D}(\mathbb{R}^d)\,.$$

This inequality generalizes the one in Subsection 3.1.3 which corresponds to $d=2$ and in particular we have $\mathsf{C}_{p,2}=\mathsf{C}_p$. Starting from (4.2), the proof by mass transportation goes as follows. An integration by parts shows that

$$\begin{aligned}\int_{\mathbb{R}^d}G^{1-\frac1d}\,\mathrm{d}y&\le-\frac{p\,(d-1)}{d\,(d-p)}\int_{\mathbb{R}^d}\nabla(F^{\frac1p-\frac1d})\cdot F^{\frac{1}{p'}}\,\nabla\phi\,\mathrm{d}x\\&\le\frac{p\,(d-1)}{d\,(d-p)}\,\|\nabla f\|_{\mathrm{L}^p(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}F\,|\nabla\phi|^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}}\,,\end{aligned}$$

where the last line relies on Hölder's inequality and the fact that $F^{\frac{1}{p}-\frac{1}{d}} = f$. The conclusion of the proof arises from the fact that $\int_{\mathbb{R}^d} F\,|\nabla\phi|^{p'}\,\mathrm{d}x = \int_{\mathbb{R}^d} G\,|y|^{p'}\,\mathrm{d}y$. It allows to characterize $\mathsf{C}_{p,d}$ by

$$\mathsf{C}_{p,d} = \frac{p\,(d-1)}{d\,(d-p)}\,\inf \frac{\left(\int_{\mathbb{R}^d} G\,|y|^{p'}\,\mathrm{d}y\right)^{\frac{1}{p'}}}{\int_{\mathbb{R}^d} G^{1-\frac{1}{d}}\,\mathrm{d}y}\,,$$

where the infimum is taken on all positive probability measures and is achieved by $G = f_\star^{\frac{d\,p}{d-p}}$. Here $f_\star(x) = (1+|x|^{p'})^{-\frac{d-p}{p}}$ is the optimal Aubin-Talenti function.

If we specialize in the case $d=2$ and consider $f = f_\star\left(1+\frac{2-p}{2\,p}\,(u-\overline{u})\right)$, where $\overline{u}$ is adjusted so that $\|f\|_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^2)} = 1$, then we recover Onofri inequality by passing to the limit as $p\to 2^-$. Moreover, we may notice that $\nabla(F^{\frac{1}{p}-\frac{1}{d}})\cdot F^{\frac{1}{p'}}\,\nabla\phi$ formally approaches $\nabla\log F\cdot\sqrt{F}\,\nabla\phi$, so that the mass transportation method for critical Sobolev inequalities is consistent with the formal computation of Subsection 4.1.

5 An Improved Inequality Based on Legendre's Duality and the Logarithmic Diffusion or Super-Fast Diffusion Equation

In [39, Theorem 2], it was shown that

$$\begin{aligned}&\int_{\mathbb{R}^2} f\,\log\left(\frac{f}{M}\right)\mathrm{d}x - \frac{4\,\pi}{M}\int_{\mathbb{R}^2} f\,(-\Delta)^{-1}\,f\,\mathrm{d}x + M\,(1+\log\pi)\\ &\le M\left[\frac{1}{16\,\pi}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} u\,\mathrm{d}\mu - \log M\right]\end{aligned}\tag{5.1}$$

holds for any function $u\in\mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} \mathrm{e}^{\,u}\,\mathrm{d}\mu$ and $f = \mathrm{e}^u\mu$. The left-hand side of (5.1) is nonnegative by the logarithmic Hardy-Littlewood-Sobolev type of inequality according to [19, Theorem 1] (also see [8, Theorem 2]). (5.1) is proven by simply expanding the square

$$0 \le \int_{\mathbb{R}^2}\left|\frac{1}{8\,\pi}\,\nabla u + \kappa\,\nabla\,(-\Delta)^{-1}(v-\mu)\right|^2\mathrm{d}x$$

for some constant κ to be appropriately chosen. Alternatively, we may work on the sphere. Let us expand the square

$$0 \le \int_{\mathbb{S}^2}\left|\frac{1}{2}\,\nabla(u-\overline{u}) + \frac{1}{\overline{v}}\,\nabla\,(-\Delta)^{-1}(v-\overline{v})\right|^2\mathrm{d}\sigma\,.$$

It is then straightforward to see that

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{S}^2}|\nabla u|^2\,\mathrm{d}\sigma + \int_{\mathbb{S}^2} u\,\mathrm{d}\sigma - \log\left(\int_{\mathbb{S}^2}\mathrm{e}^u\,\mathrm{d}\sigma\right)\\ &+\frac{1}{\overline{v}^2}\int_{\mathbb{S}^2}(v-\overline{v})\,(-\Delta)^{-1}(v-\overline{v})\,\mathrm{d}\sigma - \frac{1}{\overline{v}}\int_{\mathbb{S}^2} v\,\log\left(\frac{v}{\overline{v}}\right)\mathrm{d}\sigma\\ \ge\ &\frac{2}{\overline{v}}\int_{\mathbb{S}^2}(u-\overline{u})\,(v-\overline{v})\,\mathrm{d}\sigma + \int_{\mathbb{S}^2} u\,\mathrm{d}\sigma - \log\left(\int_{\mathbb{S}^2}\mathrm{e}^u\,\mathrm{d}\sigma\right) - \frac{1}{\overline{v}}\int_{\mathbb{S}^2} v\,\log\left(\frac{v}{\overline{v}}\right)\mathrm{d}\sigma =: \mathcal{R}[u,v]\,.\end{aligned}$$

Here we assume that

$$\overline{u} := \log\left(\int_{\mathbb{S}^2}\mathrm{e}^u\,\mathrm{d}\sigma\right)\quad\text{and}\quad \overline{v} := \int_{\mathbb{S}^2} v\,\mathrm{d}\sigma\,.$$

With the choice

$$v = e^u\,, \quad \overline{v} = e^{\overline{u}}\,,$$

the reader is invited to check that $\mathcal{R}[u,v]=0$. Altogether, we have shown that

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\log\Big(\int_{\mathbb{S}^2}e^u\,d\sigma\Big)\\ \geq&\int_{\mathbb{S}^2}f\,\log f\,d\sigma-\int_{\mathbb{S}^2}(f-1)\,(-\Delta)^{-1}(f-1)\,d\sigma\end{aligned}$$

with $f := \frac{e^u}{\int_{\mathbb{S}^2}e^u\,d\sigma}$. This inequality is exactly equivalent to (5.1). Notice that the right-hand side is nonnegative by the logarithmic Hardy-Littlewood-Sobolev inequality, which is the dual inequality of Onofri (see [19, 30, 39] for details).

Keeping track of the square, we arrive at the following identity.

Proposition 5.1 *For any $u \in H^1(\mathbb{S}^2)$, we have*

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\log\Big(\int_{\mathbb{S}^2}e^u\,d\sigma\Big)\\ =&\int_{\mathbb{S}^2}f\,\log f\,d\sigma-\int_{\mathbb{S}^2}(f-1)\,(-\Delta)^{-1}(f-1)\,d\sigma\\ &+\int_{\mathbb{S}^2}\Big|\frac{1}{2}\nabla u+\nabla\,(-\Delta)^{-1}(f-1)\Big|^2\,d\sigma\end{aligned}$$

with $f := \frac{e^u}{\int_{\mathbb{S}^2}e^u\,d\sigma}$.

It is an open question to get an improved inequality compared with (5.1) by using a flow, as was done in [39] for Sobolev and Hardy-Littlewood-Sobolev inequalities. We may, for instance, consider the logarithmic diffusion equation, which is also called the super-fast diffusion equation, on the two-dimensional sphere $\mathbb{S}^2$

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2}\log f\,, \tag{5.2}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^2$. In dimension $d=2$, (5.2) plays a role which is the analogue of the Yamabe flow in dimensions $d\geq 3$ or to be precise, of the equation $\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2} f^{\frac{d-2}{d+2}}$ (see [30, 39] for details). The flow defined by (5.2) does not give straightforward estimates, though we may notice that

$$\mathsf{H} := \int_{\mathbb{S}^2}f\,\log f\,d\sigma-\int_{\mathbb{S}^2}(f-1)\,(-\Delta)^{-1}(f-1)\,d\sigma$$

is such that, if $f = e^{\frac{u}{2}}$ is a solution to (5.2) such that $\int_{\mathbb{S}^2}f\,d\sigma = 1$, then

$$\begin{aligned}\frac{d\mathsf{H}}{dt}&=-\Big[\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\int_{\mathbb{S}^2}u\,e^{\frac{u}{2}}\,d\sigma\Big]\\ &\leq-\Big[\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\log\Big(\int_{\mathbb{S}^2}e^u\,d\sigma\Big)\Big]\end{aligned}$$

because $\int_{\mathbb{S}^2}u\,e^{\frac{u}{2}}\,d\sigma \leq \log\big(\int_{\mathbb{S}^2}e^u\,d\sigma\big)$ according to [30, Proposition 3.1].

6 An Improved Inequality Based on the Entropy-Entropy Production Method and the Fast Diffusion Equation

In $\mathbb{R}^2$, we consider the fast diffusion equation written in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot [v\,(\nabla v^{m-1} - 2\,x)] = 0\,, \tag{6.1}$$

where the parameter m is taken in the interval $[\frac{1}{2}, 1)$. According to [28], the mass $M = \int_{\mathbb{R}^2} v\,\mathrm{d}x$ is independent of t. Stationary solutions are the so-called Barenblatt profiles

$$v_\infty(x) := (D + |x|^2)^{\frac{1}{m-1}}\,,$$

where D is a positive parameter which is uniquely determined by the mass condition $M = \int_{\mathbb{R}^2} v_\infty\,\mathrm{d}x$. The relative entropy is defined by

$$\mathcal{E}[v] := \frac{1}{m-1}\int_{\mathbb{R}^2} [v^m - v_\infty^m - m\,v_\infty^{m-1}\,(v - v_\infty)]\,\mathrm{d}x\,.$$

According to [28], it is a Lyapunov functional, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[v] = -\,\mathcal{I}[v]\,,$$

where $\mathcal{I}$ is the relative Fisher information defined by

$$\mathcal{I}[v] := \int_{\mathbb{R}^2} v\,|v^{m-1} - v_\infty^{m-1}|^2\,\mathrm{d}x\,,$$

and for $m > \frac{1}{2}$, the inequality

$$\mathcal{E}[v] \le \frac{1}{4}\,\mathcal{I}[v] \tag{6.2}$$

is equivalent to a Gagliardo-Nirenberg inequality written with an optimal constant according to [28]. Note that for $m = \frac{1}{2}$,

$$v_\infty(x) := (D + |x|^2)^{-2},$$

so $v_\infty^m \notin \mathrm{L}^1(\mathbb{R}^2)$ and $|x|^2\,v_\infty \notin \mathrm{L}^1(\mathbb{R}^2)$.

However, we may consider $w = \frac{v}{v_\infty}$ at least for a function v such that $v - v_\infty$ is compactly supported, take the limit $m \to \frac{1}{2}$ and argue by density to prove that

$$\mathcal{E}[w\,v_\infty] =: E[w] = \int_{\mathbb{R}^2} \frac{|\sqrt{w} - 1|^2}{D + |x|^2}\,\mathrm{d}x \le \frac{1}{4}\,I[w]\,,$$

where

$$I[w] := \mathcal{I}[w\,v_\infty] = \int_{\mathbb{R}^2} v_\infty\,w\,\left|\nabla\big(v_\infty^{m-1}\,(w^{m-1} - 1)\big)\right|^2\mathrm{d}x$$

can be rewritten as

$$
\begin{aligned}
I[w] &= \int_{\mathbb{R}^2} \frac{w}{(D+|x|^2)^2}\,|\nabla(v_\infty^{m-1}\,(w^{m-1}-1))|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2} \frac{w}{(D+|x|^2)^2}|\nabla((D+|x|^2)\,(w^{-\frac12}-1))|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2} \frac{1}{(D+|x|^2)^2}\Big|\,2\,x\,(1-\sqrt{w}\,)-\frac12\,(D+|x|^2)\,\nabla\log w\Big|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2} \frac{4\,|x|^2\,(1-\sqrt{w}\,)^2}{(D+|x|^2)^2}\,\mathrm{d}x+\frac14\int_{\mathbb{R}^2}|\nabla\log w|^2\,\mathrm{d}x\\
&\quad -2\int_{\mathbb{R}^2} x\cdot\frac{\nabla\log w+2\,\nabla(1-\sqrt{w}\,)}{D+|x|^2}\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2} \frac{4\,|x|^2\,(1-\sqrt{w}\,)^2}{(D+|x|^2)^2}\,\mathrm{d}x+\frac14\int_{\mathbb{R}^2}|\nabla\log w|^2\,\mathrm{d}x\\
&\quad +4\,D\int_{\mathbb{R}^2}\frac{\log w+2\,(1-\sqrt{w}\,)}{(D+|x|^2)^2}\,\mathrm{d}x\;,
\end{aligned}
$$

where we performed an integration by parts in the last line. Collecting terms and letting $u=\log w$, we arrive at

$$
\begin{aligned}
\frac14\,I[w]-E[w] &= -\,D\int_{\mathbb{R}^2}\frac{(1-\sqrt{w}\,)^2}{(D+|x|^2)^2}\,\mathrm{d}x+\frac{1}{16}\int_{\mathbb{R}^2}|\nabla\log w|^2\,\mathrm{d}x\\
&\quad +D\int_{\mathbb{R}^2}\frac{\log w-2\,(\sqrt{w}-1)}{(D+|x|^2)^2}\,\mathrm{d}x\\
&= -D\int_{\mathbb{R}^2}\frac{(1-\mathrm{e}^{\frac{u}{2}})^2}{(D+|x|^2)^2}\,\mathrm{d}x+\frac{1}{16}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\\
&\quad +D\int_{\mathbb{R}^2}\frac{u-2\,(\mathrm{e}^{\frac{u}{2}}-1)}{(D+|x|^2)^2}\,\mathrm{d}x\\
&= \frac{1}{16}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x-D\int_{\mathbb{R}^2}\frac{\mathrm{e}^u-1-u}{(D+|x|^2)^2}\,\mathrm{d}x\;,
\end{aligned}
$$

and thus prove that (6.2) written for $m=\frac12$ shows that the right-hand side of the above identity is nonnegative. As a special case consider $D=1$ and define $\mathrm{d}\mu=\mu(x)\,\mathrm{d}x$ where $\mu(x)=\frac{1}{\pi}\,(1+|x|^2)^{-2}$. (6.2) can therefore be written as

$$
\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\geq\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu-1-\int_{\mathbb{R}^2}u\,\mathrm{d}\mu\,.
$$

Since $z-1\geq\log z$ for any $z>0$, this inequality implies the Onofri inequality (1.1), namely,

$$
\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\geq\log\Big(\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu\Big)-\int_{\mathbb{R}^2}u\,\mathrm{d}\mu\,.
$$

The two inequalities are actually equivalent since the first one is not invariant under a shift by a given constant: If we replace u by $u+c$ with c such that

$$
\int_{\mathbb{R}^2}e^u\,\mathrm{d}\mu-1-\int_{\mathbb{R}^2}u\,\mathrm{d}\mu\geq\mathrm{e}^c\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu-1-\int_{\mathbb{R}^2}u\,\mathrm{d}\mu-c,
$$

and minimize the right-hand side with respect to c, we get that $c=-\log(\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu)$ and recover the standard form (1.1) of Onofri inequality.

Various methods are available for proving (6.2). The Bakry-Emery method, or the carré du champ method, was developed in [2, 7] in the linear case and later extended to nonlinear diffusions in [21–22, 28] using a relative entropy which appears first in [59, 63]. This entropy-entropy production method has the advantage of providing an integral remainder term. Here we adopt a setting that can be found in [40].

Let us consider a solution v to (6.1) and define

$$z(x,t) := \nabla v^{m-1} - 2\,x\;,$$

so that (6.1) can be rewritten for any $m \in [\frac{1}{2}, 1)$ as

$$\frac{\partial v}{\partial t} + \nabla\cdot(v\,z) = 0\,.$$

A tedious computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} v\,|z|^2\,\mathrm{d}x + 4\int_{\mathbb{R}^2} v\,|z|^2\,\mathrm{d}x = -2\,\frac{1-m}{m}\,\mathcal{R}[v,z]$$

with

$$R[v,z] := \int_{\mathbb{R}^2} v^m\,[\,|\nabla z|^2 - (1-m)\,(\nabla\cdot z)^2]\,\mathrm{d}x\;, \tag{6.3}$$

where $|\nabla z|^2 = \sum_{i,j=1,2}\big(\frac{\partial z_i}{\partial x_j}\big)^2$ and $\nabla\cdot z = \sum_{i=1,2}\frac{\partial z_i}{\partial x_i}$. Summarizing, when $m = \frac{1}{2}$, we have shown that

$$\frac{1}{4}\,I[w(t=0,\cdot)] - E[w(t=0,\cdot)] = 2\int_0^\infty \mathcal{R}[v(t,\cdot), z(t,\cdot)]\,\mathrm{d}t\,.$$

Proposition 6.1 *If we denote by v the solution to* (6.1) *with an initial datum*

$$v_{|t=0} = \frac{\mathrm{e}^u}{(1+|x|^2)^2}\;,$$

then we have the identity

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} u\,\mathrm{d}\mu - \log\Big(\int_{\mathbb{R}^2}\mathrm{e}^u\,\mathrm{d}\mu\Big) = 2\int_0^\infty \mathcal{R}[v,z]\,\mathrm{d}t$$

with $\mathcal{R}$ defined by (6.3) *and $z(t,x) = \nabla v^{-\frac{1}{2}}(t,x) - 2\,x$.*

Notice that the kernel of $\mathcal{R}$ is spanned by all Barenblatt profiles, which are the stationary solutions of (6.1) (one has to take into account the invariances: Multiplication by a constant, translation and dilation). This has to do with the conformal transformation on the sphere (see Theorem 7.1 and [47, Subsection 17.3] for more details).

As a straightforward consequence of Propostion 6.1, we have the following corollary.

Corollary 6.1 *With the notations of Section* 3.2 *we have*

$$\mathcal{I}_1 = 1\,.$$

Moreover, any minimizing sequence converges to a function in the kernel of $\mathcal{R}$.

The fact that Onofri inequality is intimately related with the fast diffusion equation (6.1) with $m = \frac{1}{2}$ sheds a new light on the role played by this equation for the dual inequality, the logarithmic Hardy-Littlewood-Sobolev inequality, which was studied in [17] and applied to the critical parabolic-elliptic Keller-Segel model in [11, 18].

7 Rigidity (or Carré du Champ) Methods and Adapted Nonlinear Diffusion Equations

By rigidity method, we refer to a method which was popularized in [48] and optimized later in [10]. We will first consider the symmetric case in which computations can be done along the lines of [32] and are easy. Then we will introduce flows as in [32] (for Sobolev inequality and interpolation inequalities in the subcritical range), still in the symmetric case. The main advantage is that the flow produces an integral remainder term which is, as far as we know, a new result in the case of Onofri inequality.

The integrations by parts of the rigidity method can be encoded in the Γ_2 or carré du champ methods, thus providing the same results. In the case of Onofri inequality, this has been observed by Fontenas in [43, Theorem 2] (actually, without symmetry).

A striking observation is indeed that no symmetry is required. The rigidity computations and the flow can be used in the general case, as was done in [36], and produce an integral remainder term, which is our last new result.

7.1 The rigidity method in the symmetric case

As shown for instance in [62] the functional

$$\mathcal{G}_\lambda[v] := \frac{1}{4}\int_{\mathbb{S}^2} |\nabla v|^2\,\mathrm{d}\sigma + \lambda\int_{\mathbb{S}^2} v\,\mathrm{d}\sigma - \log\Big(\int_{\mathbb{S}^2} \mathrm{e}^v\,\mathrm{d}\sigma\Big)$$

is nonnegative for all $\lambda > 0$ and it can be minimized in $\mathrm{H}^1(\mathbb{S}^2)$ and, up to the addition of a constant, any minimizer satisfies the Euler-Lagrange equation

$$-\frac{1}{2}\,\Delta v + \lambda = \lambda \mathrm{e}^v \quad \text{on } S^2\,. \tag{7.1}$$

According to Proposition 3.2, minimizing $\mathcal{G}_\lambda$ amounts to minimizing

$$\mathsf{G}_\lambda[f] := \frac{1}{8}\int_{-1}^1 |f'|^2\,\nu\,\mathrm{d}z + \frac{\lambda}{2}\int_{-1}^1 f\,\mathrm{d}z \ge \lambda\log\Big(\frac{1}{2}\int_{-1}^1 \mathrm{e}^f\,\mathrm{d}z\Big)\,,$$

and (3.9) can be reduced to the fact that the minimum of G_1 is achieved by constant functions. For the same reasons as above, G_λ has a minimum which solves the Euler-Lagrange equation

$$-\frac{1}{2}\,\mathcal{L}\,f + \lambda = 2\lambda\frac{\mathrm{e}^f}{\int_{-1}^1 \mathrm{e}^f\,\mathrm{d}z}\,,$$

where $\mathcal{L}\,f := \nu\,f'' + \nu'\,f'$ and $\nu(z) = 1 - z^2$. Up to the addition of a constant, we may choose f such that $\int_{-1}^1 \mathrm{e}^f\,\mathrm{d}z = 2$ and hence solves

$$-\frac{1}{2}\,\mathcal{L}\,f + \lambda = \lambda \mathrm{e}^f\,. \tag{7.2}$$

Theorem 7.1 *For any $\lambda \in (0,1)$, (7.2) has a unique smooth solution f, which is the constant function*

$$f = 0.$$

As a consequence, if f is a critical point of the functional G_λ, then f is a constant function for any $\lambda \in (0,1)$, while for $\lambda = 1$, f has to satisfy the differential equation $f'' = \frac{1}{2}\,|f'|^2$ and is either a constant, or such that

$$f(z) = C_1 - 2\,\log(C_2 - z) \tag{7.3}$$

for some constants $C_1 \in \mathbb{R}$ and $C_2 > 1$.

Let us define

$$\mathsf{R}_\lambda[f] := \frac{1}{8} \int_{-1}^{1} \nu^2 \left| f'' - \frac{1}{2} |f'|^2 \right|^2 e^{-\frac{f}{2}} \, dz + \frac{1-\lambda}{4} \int_{-1}^{1} \nu \, |f'|^2 \, e^{-\frac{f}{2}} \, dz \,. \tag{7.4}$$

The proof is a straightforward consequence of the following lemma.

Lemma 7.1 *If f solves* (7.2)*, then*

$$\mathsf{R}_\lambda[f] = 0 \,.$$

Proof The ultraspherical operator does not commute with the derivation with respect to z:

$$(\mathcal{L} \, f)' = \mathcal{L} \, f' - 2 \, z \, f'' - 2 \, f' \,,$$

where $f' = \frac{df}{dz}$. After multiplying (7.2) by $\mathcal{L} \, (e^{-\frac{f}{2}})$ and integrating by parts, we get

$$\begin{aligned} 0 &= \int_{-1}^{1} \Big(-\frac{1}{2} \, \mathcal{L} \, f + \lambda - e^{f} \Big) \, \mathcal{L} \, (e^{-\frac{f}{2}}) \, dz \\ &= \frac{1}{4} \int_{-1}^{1} \nu^2 \, |f''|^2 \, e^{-\frac{f}{2}} \, dz - \frac{1}{8} \int_{-1}^{1} \nu^2 \, |f'|^2 \, f'' \, e^{-\frac{f}{2}} \, dz \\ &\quad + \frac{1}{2} \int_{-1}^{1} \nu \, |f'|^2 \, e^{-\frac{f}{2}} \, dz - \frac{1}{2} \int_{-1}^{1} \nu \, |f'|^2 \, e^{\frac{f}{2}} \, dz \,. \end{aligned}$$

Similarly, after multiplying (7.2) by $\frac{\nu}{2} \, |f'|^2 \, e^{-\frac{f}{2}}$ and integrating by parts, we get

$$\begin{aligned} 0 &= \int_{-1}^{1} \Big(-\frac{1}{2} \, \mathcal{L} \, f + \lambda - e^{f} \Big) \Big(\frac{\nu}{2} \, |f'|^2 \, e^{-\frac{f}{2}} \Big) \, dz \\ &= \frac{1}{8} \int_{-1}^{1} \nu^2 \, |f'|^2 \, f'' \, e^{-\frac{f}{2}} \, dz - \frac{1}{16} \int_{-1}^{1} \nu^2 \, |f'|^4 \, e^{-\frac{f}{2}} \, dz \\ &\quad + \frac{\lambda}{2} \int_{-1}^{1} \nu \, |f'|^2 \, e^{-\frac{f}{2}} \, dz - \frac{1}{2} \int_{-1}^{1} \nu \, |f'|^2 \, e^{\frac{f}{2}} \, dz \,. \end{aligned}$$

Subtracting the second identity from the first, one establishes the first part of the theorem. If $\lambda \in (0,1)$, then f has to be a constant. If $\lambda = 1$, there are other solutions, because of the conformal transformations (see for instance [47, Subsection 17.3] for more details). In our case, all solutions of the differential equation $f'' = \frac{1}{2} \, |f'|^2$ that are not constant are given by (7.3).

7.2 A nonlinear flow method in the symmetric case

Consider the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = \mathcal{L} \, (e^{-\frac{g}{2}}) - \frac{\nu}{2} \, |g'|^2 \, e^{-\frac{g}{2}} \,. \tag{7.5}$$

Proposition 7.1 *Assume that g is a solution to* (7.5) *with an initial datum $f \in \mathrm{L}^1(-1,1;dz)$ such that $\int_{-1}^{1} |f'|^2 \, \nu \, dz$ is finite and $\int_{-1}^{1} e^{f} \, dz = 1$. Then for any $\lambda \in (0,1]$, we have*

$$\mathsf{G}_\lambda[f] \geq \int_0^\infty \mathsf{R}_\lambda[g(t,\cdot)] \, dt \,,$$

where R_λ is defined in (7.4)*.*

Proof A standard regularization method allows us to reduce the evolution problem to the case of smooth bounded functions, at least at a finite-time interval. Then a simple computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{G}_\lambda[g(t,\cdot)] = -\frac{1}{2}\int_{-1}^{1}\Big(-\frac{1}{2}\,\mathcal{L}\,g+\lambda-\mathrm{e}^{g}\Big)\,\frac{\partial g}{\partial t}\,\mathrm{d}z = -\mathsf{R}_\lambda[g(t,\cdot)]\,.$$

We may then argue by continuation. Because $\mathsf{G}_\lambda[g(t,\cdot)]$ is bounded from below, $\mathsf{R}_\lambda[g(t,\cdot)]$ is integrable with respect to $t \in [0,\infty)$. Hence, as $t \to \infty$, g converges to a constant if $\lambda < 1$, or to the conformal transformation of a constant if $\lambda = 1$ and therefore $\lim_{t\to\infty} \mathsf{G}_\lambda[g(t,\cdot)] = 0$. The result holds with equality after integrating on $[0,\infty) \ni t$. For a general initial datum without the smoothness assumption, we conclude by density and get an inequality instead of an equality by lower semi-continuity.

For a general function $v \in \mathrm{H}^1(\mathbb{S}^2)$, if we denote by v_* the symmetrized function which depends only on θ (see [47, Section 17.1] for more details) and denote by f the function such that $f(\cos\theta) = v_*(\theta)$, then it follows from Propositions 3.2 and 7.1 that

$$\mathcal{G}_\lambda[v] \geq \int_0^\infty \mathsf{R}_\lambda[g(t,\cdot)]\,\mathrm{d}t\;,$$

where g is the solution to (7.5) with an initial datum f. However, we do not need any symmetrization step, as we shall see in the next section.

7.3 A nonlinear flow method in the general case

On $\mathbb{S}^2$ let us consider the nonlinear evolution equation

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2}\,(\mathrm{e}^{-\frac{f}{2}}) - \frac{1}{2}\,|\nabla f|^2\,\mathrm{e}^{-\frac{f}{2}}\;, \tag{7.6}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator. Let us define

$$\mathcal{R}_\lambda[f] := \frac{1}{2}\int_{\mathbb{S}^2}\Big\|\mathrm{L}_{\mathbb{S}^2} f - \frac{1}{2}\,\mathrm{M}_{\mathbb{S}^2} f\Big\|^2\,\mathrm{e}^{-\frac{f}{2}}\,\mathrm{d}\sigma + \frac{1}{2}\,(1-\lambda)\int_{\mathbb{S}^2}|\nabla f|^2\,\mathrm{e}^{-\frac{f}{2}}\,\mathrm{d}\sigma\;,$$

where

$$\mathrm{L}_{\mathbb{S}^2} f := \mathrm{Hess}_{\mathbb{S}^2} f - \frac{1}{2}\,\Delta_{\mathbb{S}^2} f\,\mathrm{Id} \quad\text{and}\quad \mathrm{M}_{\mathbb{S}^2} f := \nabla f \otimes \nabla f - \frac{1}{2}\,|\nabla f|^2\,\mathrm{Id}\,.$$

This definition of $\mathcal{R}_\lambda$ generalizes the definition of R_λ given in Subsection 7.1 in the symmetric case. We refer to [36] for more detailed considerations, and to [32] for considerations and improvements of the method that are specific to the sphere $\mathbb{S}^2$.

Theorem 7.2 *Assume that f is a solution to* (7.6) *with an initial datum* $v - \log\big(\int_{\mathbb{S}^2}\mathrm{e}^{v}\,\mathrm{d}\sigma\big)$, *where $v \in \mathrm{L}^1(\mathbb{S}^2)$ is such that $\nabla v \in \mathrm{L}^2(\mathbb{S}^2)$. Then for any $\lambda \in (0,1]$ we have*

$$\mathcal{G}_\lambda[v] \geq \int_0^\infty \mathcal{R}_\lambda[f(t,\cdot)]\,\mathrm{d}t\,.$$

Proof With no restriction, we may assume that $\int_{\mathbb{S}^2}\mathrm{e}^{v}\,\mathrm{d}\sigma = 1$ and it is then straightforward to see that $\int_{\mathbb{S}^2}\mathrm{e}^{f(t,\cdot)}\,\mathrm{d}\sigma = 1$ for any $t > 0$. Next we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_\lambda[f] = \int_{\mathbb{S}^2}\Big(-\frac{1}{2}\,\Delta_{\mathbb{S}^2} f + \lambda\Big)\Big(\Delta_{\mathbb{S}^2}\,(\mathrm{e}^{-\frac{f}{2}}) - \frac{1}{2}\,|\nabla f|^2\,\mathrm{e}^{-\frac{f}{2}}\Big)\,\mathrm{d}\sigma = -\,\mathcal{R}_\lambda[f]$$

in the same spirit as that in [36].

As a concluding remark, let us notice that the carré du champ method is not limited to the case of $\mathbb{S}^2$, but also applies to two-dimensional Riemannian manifolds (see for instance [42]). The use of the flow defined by (7.6) gives an additional integral remainder term, in the spirit of what was done in [36]. This is, however, out of the scope of the present paper.

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